PHYSICS Ph.D. QUALIFYING EXAMINATION – PART A

Tuesday, August 20, 2024, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put **your identifying number on each page**. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some may find useful the Schaum's outline, '*Mathematical Handbook of Formulas and Tables.*'

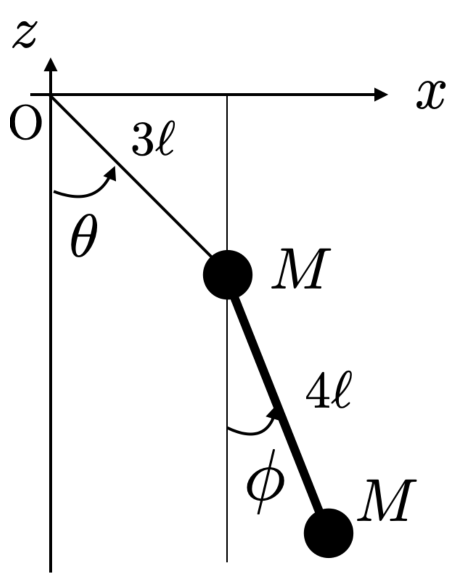
A1. Classical Mechanics

As shown in the figure below, two particles with mass M are connected with a massless bar with length 4ℓ, and one of them and the origin are connected with a massless taut string with length of 3ℓ. The only force to consider is gravity with a constant acceleration of g directed towards negative z. Consider two angle variables, (θ, ϕ), as indicated in the figure.

1) Find the Lagrangian of the system.

2) Write down the Lagrange equation of motions, and find the angular eigen frequencies for small angles, i.e., $\theta \ll 1$ and $\phi \ll 1$, in terms of $\omega_0 \equiv \sqrt{g/\ell}$.

3) Suppose that the system is initially at rest, $\theta(t = 0) = \phi(t = 0) = 0$. At time $t = 0$, an impulsive force acts on the bar at a point with z coordinate, $z = h - 5l$. You observe that only the lower of the two eigen frequencies gets exited. Find h in terms of ℓ. [Hint: consider how the linear and angular momentum about the center of the bar are changed by the impulsive force.]



A2. Classical Mechanics

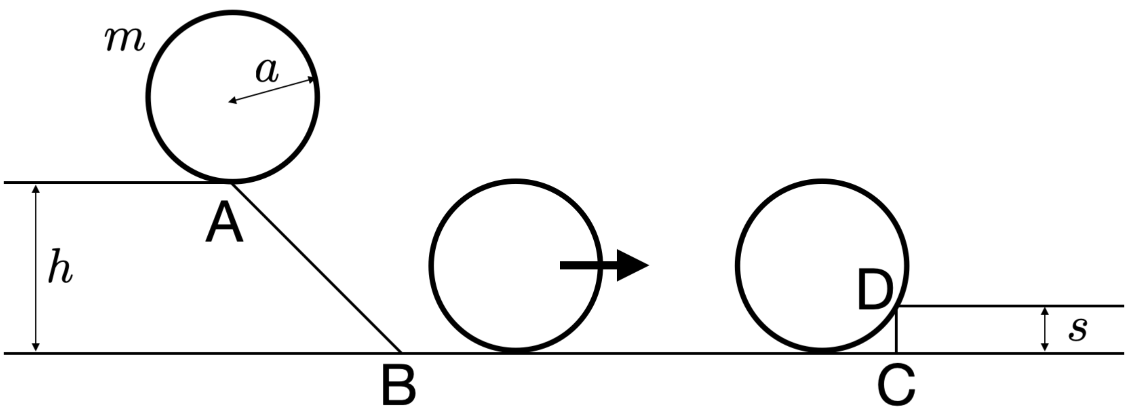
As shown in the figure below, a uniform cylinder with radius a and mass m starts at rest at the point A (with height h), rolls down the slope to the point B, passes a flat horizontal surface between B and C, and climbs up the step CD with height s without bounce where $s < a$. Assume that the cylinder rotates without slip throughout its motion and does not lose any energy from friction.

1) Find the moment of inertia of the uniform cylinder with respect to the rotation axis between A and C.

2) Find the angular speed of the cylinder while it passes BC.

3) Find the angular speed of the cylinder *immediately* after the cylinder collides with the point D. You may assume that the cylinder rotates around D with no slip and that the angular momentum around the point D throughout this process is conserved.

4) Find the minimum height h so that the cylinder can climb up the step CD.



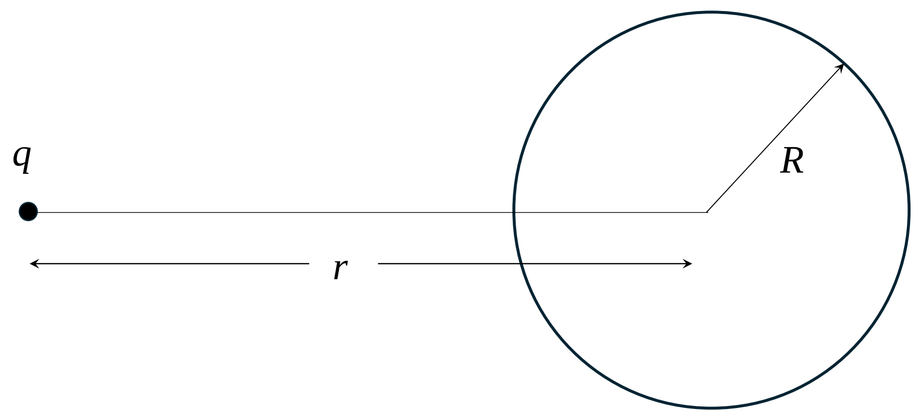
A3. Electrodynamics

A positive charge q is located at a distance r from the center of a conducting sphere of radius R. The sphere and charge do not move.

1) If the sphere is electrically grounded, then a charge $q' = -q(R/r)$ is induced on the sphere. Find the position d of the image charge q'. Use d as a parameter to solve the next two questions.

2) Find the magnitude and direction of the force exerted on the sphere by the charge q.

3) If the sphere is not grounded but is a charged, insulated, conducting sphere, what is the total charge Q that it must carry in order for the total force on the sphere to be zero? [Hint: Use superposition of the induced and net charges.]



A4. Electrodynamics

A spherical charge distribution is given by

$$\rho = \rho_0(1 - \frac{r^2}{a^2}), \qquad r < a$$
$$\rho = 0, \qquad r > a$$

1) Calculate the total charge Q.

2) Find the electric field strength $\vec{E}$ and the potential V outside the charge distribution. Assume $V = 0$ at $r = \infty$.

3) Find $\vec{E}$ and V inside the charge distribution.

4) Show that the maximum value of E is at $\frac{r}{a} = 0.745 = \sqrt{\frac{5}{9}}$.

A5. Quantum Mechanics

Two particles of mass μ and spin 1/2 interact only through spin-spin coupling:

$$V = -u\boldsymbol{s_1} \cdot \boldsymbol{s_2}, \quad u > 0,$$

1) Calculate the energy shift of the system due to the spin interaction for the singlet and triplet state.

2) The particles are now trapped in the one-dimensional interval $0 < x < a$ (i.e., in a box of width a). Determine the spatial part $\psi_{n,m}(x_1, x_2)$ of the eigenstate. What are the total eigenenergies (including spin-spin interaction)? Is the ground state a singlet or triplet state? Explain why.

3) A weak perturbation potential of the form

$$U(x) = \begin{cases} \delta & x > a/2 \\ -\delta & x < a/2 \end{cases}$$

is additionally applied to the system. What is the first-order correction to the eigenenergy of the ground state?

4) Consider the system being in its ground state when it undergoes a sudden (instantaneous) change of the trapping potential width from a to $b = 2a$. What is the probability p to still find the particles in their ground state after the sudden expansion of the trapping potential?

A6. Modern Physics I

The Lagrangian of a relativistic particle in 3 dimensions is given by

$$L = -mc^2\sqrt{1 - \left|\dot{\vec{q}}\right|^2/c^2},$$

where $\vec{q}$ has three components $\{q_1, q_2, q_3\}$ and the dot denotes the local time derivative.

1) Derive the expression for the relativistic momentum $\vec{p}$ of the particle and prove that it is conserved.

2) Derive the expression for the relativistic energy E of the particle.

3) Prove that $\vec{p}$ and E satisfy the following relationship

$$E^2 = \left|\dot{\vec{p}}\right|^2 c^2 + m^2c^4.$$

PHYSICS Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, August 21, 2024, 1:00 – 5:00 P.M.

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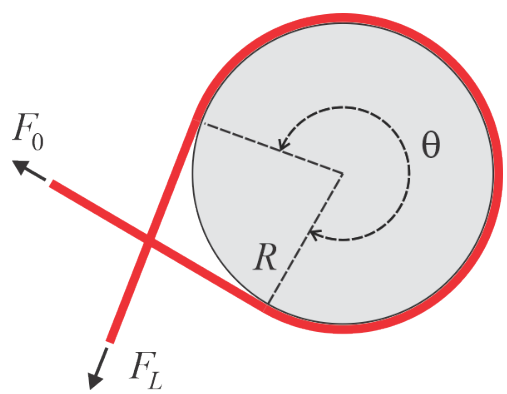
B1. Classical Mechanics

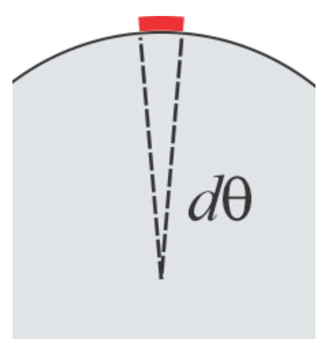
A rope under a large tension force (load) F_L is wound around a fixed cylinder of radius R, subtending an angle θ as shown in the left figure. (θ can be less than 2π as in the figure or larger than 2π if the rope makes multiple turns.) As a result of static friction between the rope and the cylinder, a smaller force F_0 is sufficient to control the load.

1) Consider the forces acting on an infinitesimal piece of the rope that subtends an angle $d\theta$, as shown in the right figure. Find the friction force acting on this piece in terms of the (uniform) friction coefficient μ and the tensions at its left and right ends.

2) Derive a differential equation for the tension as function of the angle.

3) Solve the differential equation and find an expression for the minimum value of F_0 that prevents the rope from slipping in terms of F_L, μ, and the total angle θ subtended by the rope.





B2. Electrodynamics

Consider an electromagnetic wave with a transverse amplitude profile given by $E_o(x, y) = \mathcal{A}e^{-(x^2+y^2)/(4\sigma^2)}$ propagating in vacuum along the z direction.

1) Show that the fields

$$\boldsymbol{E}^{(0)}(t, \boldsymbol{r}) = E_o(x, y)e^{i(kz-\omega t)}\frac{\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}}{\sqrt{2}} \tag{1}$$

$$\boldsymbol{B}^{(0)}(t, \boldsymbol{r}) = \frac{1}{c}\hat{\boldsymbol{z}} \times \boldsymbol{E}^{(0)}(t, \boldsymbol{r}) \tag{2}$$

(with $\omega = ck$ and $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ being the unit vectors along the Cartesian axes) are a solution of the Maxwell Equations, if the derivatives of E_o with respect to x and y are neglected.

2) What is the time averaged energy per unit length (along the propagation direction) $\langle U \rangle$ of the field.

3) If the derivatives of $E_o(x, y)$ are not neglected, Eqs. 1 and 2 do not solve the Maxwell Equations. What are the first order corrections to the electric and magnetic fields (for $k\sigma \gg 1$, i.e., for the beam waist being much larger than the wavelength).
Hint: Use $\boldsymbol{E}$ (and analogously $\boldsymbol{B}$) with a (small) z-component of the form

$$\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E}^{(0)} + E^{(1)}(x, y)e^{i(kz-\omega t)}\hat{\boldsymbol{z}}, \tag{3}$$

and find $E^{(1)}(x, y)$ in terms of $E_o(x, y)$ (and its derivatives), by exploiting the vanishing divergence of the field.

4) Calculate the z-component of the time averaged angular momentum per unit lenght $\langle L_z \rangle$ of the electromagnetic field considering the first-order corrections determined above.
Hint: Remember that the momentum density of an electromagnetic field (i.e., its linear momentum per unit volume) is given by $\sqrt{\frac{\epsilon_0}{\mu_0}}\boldsymbol{E} \times \boldsymbol{B}$.

5) Determine the ratio of angular momentum and energy $\langle L_z \rangle / \langle U \rangle$. What is this ratio in the photon picture?

B3. Statistical Mechanics

A box of volume V contains a classical ideal gas of N point particles of mass m that is kept at temperature T by a thermostat. At time $t = 0$, a small hole of cross section area A is opened in the box, and gas starts escaping. (Assume that the hole is sufficiently small so that the gas in the box remains in equilibrium during the effusion process.)

1) Starting from the Maxwell distribution, find the number of particles escaping through the hole per time as function of the system parameters N, V, A, m, and T.

2) Derive a differential equation for the pressure p in the box as a function of the time t after opening the hole.

3) Solve the differential equation and find $p(t)$ in terms of the system parameters.

B4. Quantum mechanics

A particle of mass m is constrained to move in a circle of radius R lying in the $x-y$ plane. Let the position of the particle be denoted by the angle ϕ, which is measured with respect to the positive x-axis. The classical Hamiltonian for this system is

$$H = \frac{L_z^2}{2mR^2}$$

At $t = 0$,

$$\psi(\phi, 0) = \frac{1}{\sqrt{2\pi}}(\cos\phi - \sin\phi)$$

1) Calculate the probability that a measurement of L_z at $t = 0$ will yield the result $\hbar$.

2) Obtain $\psi(x, t)$.

B5. Quantum mechanics

Consider the state described by the following normalized column vector of coefficients of energy eigenstates:

$$\psi(t) = \frac{1}{\sqrt{6}} \begin{pmatrix} e^{-iE_1 t/\hbar} \\ \sqrt{2}e^{-iE_2 t/\hbar} \\ \sqrt{3}e^{-iE_3 t/\hbar} \end{pmatrix},$$

where E_1, E_2, and E_3 are the allowed energies for the system. Take $E_1 = \lambda$, $E_2 = 2\lambda$, and $E_3 = 4\lambda$ where λ is a constant.

1) If a large number (statistical ensemble) of systems were each prepared in this state and the energy of each is measured at time t, what would the mean and standard deviation of the result be? Express your answers in terms of λ.

2) The operator $\hat{Q}$ associated with the observable Q is

$$\hat{Q} = \beta \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$$

where β is a constant. Find the possible values of the measurement of Q. What is the probability that the measurement outcome will be 0 at $t = 0$?

B6. Modern Physics II

A beam of photons with frequency ω is incident onto a solid. As a result of interaction of the photons with electrons in the solid, some electrons can be ejected. Assume that the relativistic effects can be neglected and the motion of electrons can be treated classically.

1) What is the name of the famous experiment and who is credited with explaining it?

2) Describe the interpretation of the phenomenon based on classical physics. Explain how the light was thought to interact with the electrons i.e. the process by which the energy of the incident radiation is transferred to the ejected electrons. Based on this mechanism, how does the frequency/intensity of the light affect the process?

3) Write and justify the energy conservation equation governing the quantum mechanical description of the experiment. Define the quantities entering it.

4) Describe at least two experimental observations, which are inconsistent with the classical interpretation of the experiment. In each case, describe the behavior expected based on the classical interpretation and how the quantum mechanical interpretation resolves the inconsistency.