PHYSICS Ph.D. QUALIFYING EXAMINATION – PART A

Tuesday, August 26, 2025, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put **your identifying number on each page**. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some may find useful the Schaum's outline, '*Mathematical Handbook of Formulas and Tables.*'

A1. Classical Mechanics

A rocket that initially consists of rocket mass M_r and fuel mass M_f is vertically launched from the surface of Earth at $t = 0$ to escape from the gravitational field of the Earth. Let the height from the Earth surface, velocity, and mass of the rocket at time t be $h(t)$, $v(t) = dh/dt$, and $M(t)$. The relative velocity of the fuel gas with respect to the rocket rest frame is kept constant, u. The gravitational acceleration constant and the radius of the Earth are denoted by g and R_E, respectively. You may assume that $R_E \gg h$ and $v \ll c$ where c is the speed of light.

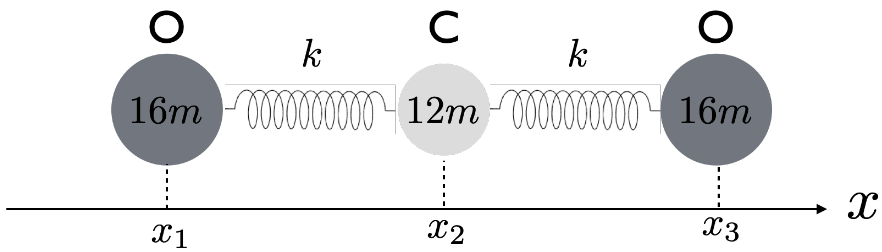
1) Show that the equation of motion of the rocket is given by

$$M\frac{dv}{dt} = -u\frac{dM}{dt} - Mg. \tag{1}$$

2) After burning all the fuel, the rocket reaches a height h_f. What is the minimum velocity v_{esc} the rocket must have at this point in order to escape Earth's gravitational field? Express your answer in terms of the given symbols. In addition, show that the minimum velocity does not depend on h_f in the limit $h_f/R_E \ll 1$.

3) Assume that the rocket releases all its fuel in a fixed time interval T. Find the resulting velocity gain Δv in terms of the given symbols. Based on your result, describe two ways to increase the velocity gain.

A2. Classical Mechanics



As shown, a CO_2 molecule is modeled by two identical springs with a spring constant k, and three atoms are constrained to move in a one-dimensional position, x. The mass of carbon and oxygen atoms is given by $12m$ and $16m$, respectively (m is a positive constant).

1) Write down the Lagrangian of the system and find the equations of motion for the three coordinates x_1, x_2, and x_3.

2) Find the position of the center of mass of the system, R, and show that the center of mass follows a constant in motion, i.e., $\ddot{R} = 0$ and thus $\dot{R}$ =constant?

3) Given 2), we set $R = dR/dt = 0$. Then find the angular frequency of the two normal modes, and briefly describe how these two oscillation modes behave. In addition, three degrees of freedom must lead to three normal modes. Discuss what the third normal mode corresponds to.

A3. Electrodynamics

A static charge distribution with charge density $\rho(r) = \frac{q_0 r_0}{r^4}$ fills the region $r > R$, where the constant q_0 has the units of charge and r_0 is a constant with dimensions of length. For the region $r < R$, $\rho = 0$.

a) What is the electric field $\vec{E}(r)$ at a distance r from the center of the sphere for $r < R$?

b) What is the electric field $\vec{E}(r)$ at a distance r from the center of the sphere for $r > R$?

c) Find the electrostatic potential V as a function of r, assuming $V = 0$ at infinity.

d) A thin hollow grounded conducting spherical shell of radius R, centered at the origin, is now added. What is the induced charge density on the inner and outer surfaces of the shell?

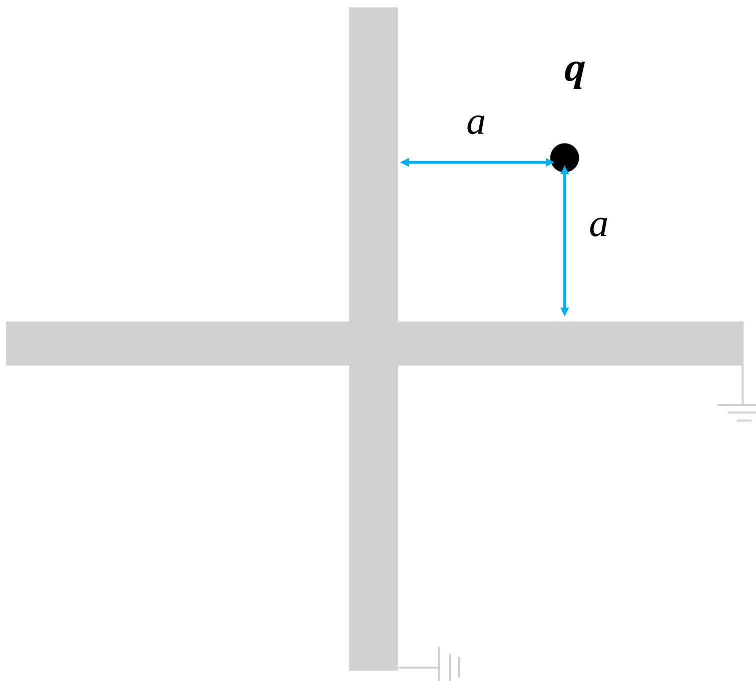
A4. Electrodynamics

A charge q is placed adjacent to two infinite grounded conducting planes as shown in the figure.

a) Determine the electrostatic potential everywhere in the first quadrant.

b) Determine the work needed to bring q to position (a, a) from infinity.

c) Determine the force on charge q.



A5. Quantum Mechanics

In this problem, we explore how the hyperfine energy levels of the hydrogen atom (in its ground state) are affected by a weak external magnetic field.

The magnetic part of the Hamiltonian for a hydrogen atom in the 1s ground state, in the presence of a constant magnetic field B in the z direction, may be written as

$$\hat{H} = B\left(\mu_{\mathrm{e}}\hat{\sigma}_z^{(\mathrm{e})} + \mu_{\mathrm{p}}\hat{\sigma}_z^{(\mathrm{p})}\right) + W\hat{\boldsymbol{\sigma}}^{(\mathrm{e})} \cdot \hat{\boldsymbol{\sigma}}^{(\mathrm{p})}$$

where the superscripts e and p refer to the electron and proton, the vector components of $\hat{\boldsymbol{\sigma}}$ are the Pauli spin operators, $\mu_{\mathrm{e,p}}$ are the respective magnetic dipole moments, and W is a constant.

1) Explain the physical origin of each term in the Hamiltonian.

2) Using a basis set given by $|{\uparrow_{\mathrm{e}}}\rangle \otimes |{\uparrow_{\mathrm{p}}}\rangle$, $|{\uparrow_{\mathrm{e}}}\rangle \otimes |{\downarrow_{\mathrm{p}}}\rangle$, $|{\downarrow_{\mathrm{e}}}\rangle \otimes |{\uparrow_{\mathrm{p}}}\rangle$, and $|{\downarrow_{\mathrm{e}}}\rangle \otimes |{\downarrow_{\mathrm{p}}}\rangle$, and neglecting the small term with μ_{p}, show that the Hamiltonian may be represented by the matrix

$$\hat{H} = \begin{pmatrix} b+W & 0 & 0 & 0 \\ 0 & b-W & 2W & 0 \\ 0 & 2W & -b-W & 0 \\ 0 & 0 & 0 & -b+W \end{pmatrix} \tag{2}$$

with $b = \mu_{\mathrm{B}}B$.

3) Determine the energy levels and sketch their evolution as a function of B, labeling them with as much information as possible about the total angular momenta of the states.

A6. Modern Physics I

A high-energy positron e^+ with energy E (in the lab reference-frame) collides head-on with a stationary electron e^-. The collision produces a muon-antimuon pair, where muons are observed at $\theta = \pm 45°$ angles relative to the incoming positron direction:

$$e^+ + e^- \to \mu^+ + \mu^-$$

a) Find analytical expression for initial energy E of the positron.

b) Using numerical values for electron mass ($m_e = 0.511$ MeV/c^2) and muon mass ($m_\mu = 105.7$ MeV/c^2), verify that initial energy of the system exceeds the rest-mass energy for the muon pair.

PHYSICS Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, August 27, 2025, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some may find useful the Schaum's outline, 'Mathematical Handbook of Formulas and Tables.'

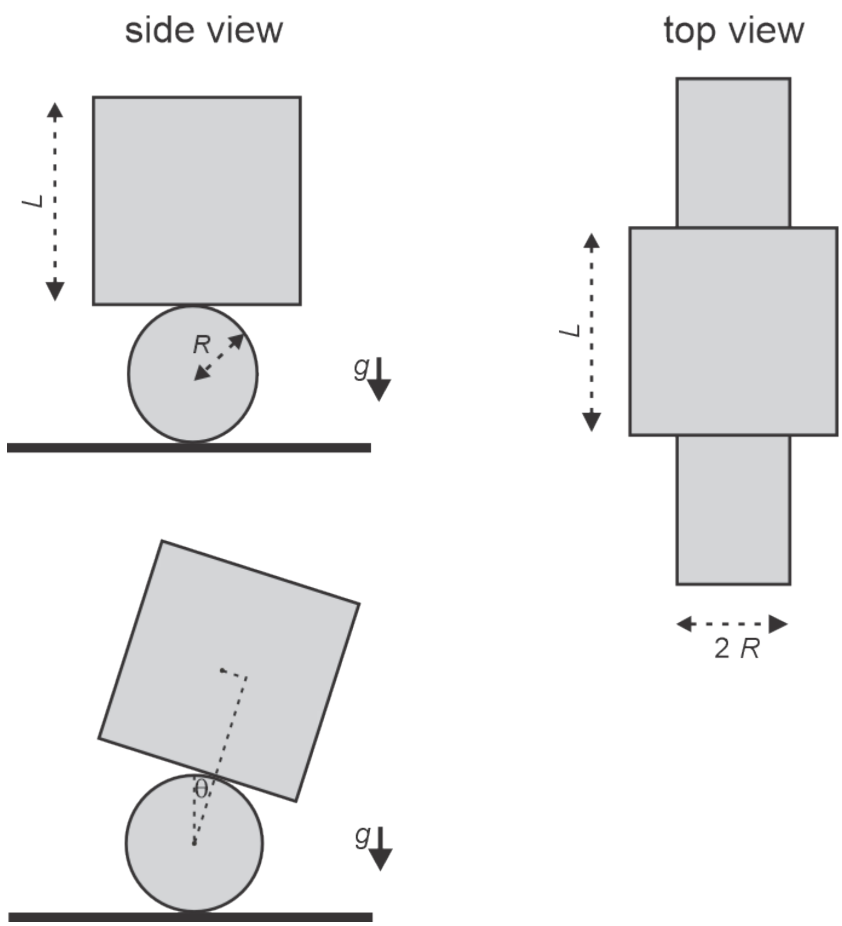
B1. Classical Mechanics

A long cylinder of radius R lies horizontally on a table. It is fixed and cannot move. A cube of side L and mass M is balanced on top of the cylinder. The cube has a uniform density and is initially oriented horizontally, with its center located exactly above the center line of the cylinder. The surfaces are rough, such that the slab cannot slide, but it can roll (rotate) from side to side, as shown in the figure.

(a) Determine the potential energy of the cube as a function of the tilt angle Θ defined in the bottom left figure.

(b) Find the values of L (in terms of R) for which the horizontal orientation of the cube, $\Theta = 0$, is a stable equilibrium position.

(c) For the cases where the horizontal position is stable, find the frequency of small oscillations of the cube about the balanced position. [Hint: The moment of inertia of the cube about an axis that goes through its center is $M\,L^2/6$]



B2. Electrodynamics

1) Starting from the Lorentz force law, show that the total work done per unit time on all charged particles in a system is given by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int \boldsymbol{E}(\boldsymbol{x}) \cdot \boldsymbol{J}(\boldsymbol{x}) \mathrm{d}^3 x,$$

where $\boldsymbol{J}(\boldsymbol{x})$ is the current density of the charged particles.

2) A constant current I flows uniformly through a straight cylindrical wire of finite resistance R, across which a potential difference V is maintained between its two ends. The wire has a length L and radius $a \ll L$. Compute the electric and magnetic fields at the surface of the wire, and compute the Poynting vector. Explicitly indicate the direction of the Poynting vector with respect to the wire.

3) Determine the electromagnetic energy entering the wire per unit time, and compare it to the Ohmic heating.

B3. Statistical Mechanics

Consider a classical ideal gas of N point particles of mass m, in thermal equilibrium at temperature T. The particles are constraint to two-dimensions, i.e., they can only move in the xy plane in a square of area A.

(a) Write down the Maxwell-Boltzmann velocity distribution for this two-dimensional ideal gas.

(b) Derive the (properly normalized) probability density of the energy $\rho(E)$ of one of the particles. ($\rho(E)dE$ is the probability that the particle has an energy between E and $E+dE$).

(c) Find the average energy per particle as a function of temperature T.

(d) What is the probability that a particle has at least 100 times the average energy?

(e) Find the number of particles that hit a segment of length ℓ of the wall of the square during a time interval Δt. Express the answer in terms of ℓ and Δt as well as N, A, m and T.

B4. Quantum mechanics

Two identical non-relativistic particles of mass m are confined to one dimension (the x axis). Each particle moves in a harmonic trapping potential

$$V(x) = \frac{1}{2}kx^2$$

where $k > 0$. The coordinate operator of a harmonic oscillator can be written in terms of raising and lowering operators as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)$$

where $\omega = \sqrt{k/m}$.

(a) If the two particles are spinless non-interacting bosons, find the ground-state energy E_0 and first excited state energy E_1 of the system.

(b) Solve part (a) for spinless non-interacting fermions.

(c) Assume the two particles are non-interacting spin-1/2 electrons. What are E_0 and E_1 and their degeneracies?

(d) For the remainder of this problem, consider an additional attractive interaction which is added to the system Hamiltonian:

$$V_{\text{int}}(x_1, x_2) = -\beta k x_1 x_2$$

where x_1 and x_2 are the particle coordinates and β is a dimensionless parameter with $0 < \beta < 1$ (parts (a)-(c) had $\beta = 0$). For spinless bosons, find the correction to E_0 to lowest non-vanishing order in β. Reminder: perturbation theory formulas for the first ($E^{(1)}$) an second ($E^{(2)}$) orders are:

$$E_n^{(1)} = \langle n^{(0)}|V|n^{(0)}\rangle$$

$$E_n^{(2)} = \sum_{m\neq n} \frac{|\langle n^{(0)}|V|m^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

where $E_n^{(0)}$ and $|n^{(0)}\rangle$ are non-perturbed eigenvalues and eigenstates.

B5. Quantum mechanics

The variational method in quantum mechanics is often accurate for the determination of ground-state energies. The key idea is that ground-state state-vector $|\psi_0\rangle$ minimizes the expectation value of the Hamiltonian operator $\hat{H}$, i.e. $\langle\psi|H|\psi\rangle > E_0 = \langle\psi_0|H|\psi_0\rangle$, where $\langle\psi|\psi\rangle = 1$ is an arbitrary (normalized) state-vector. One then guesses $|\psi\rangle$ based on physical intuition and symmetry, employing one or more free parameters, and then minimizes the $\langle\psi|H|\psi\rangle$ with respect to those parameters to determine the particular values of the parameters that give the best estimate for the ground-state energy. Note that parts (b),(c) and (d) can be done even if (a) is difficult.

(a) Show that $\langle\psi|H|\psi\rangle \geq E_0$, where E_0 is the true ground-state energy and $|\psi\rangle$ is any state-vector. Hint: begin by expanding the arbitrary state in terms of the eigenstates of the Hamiltonian.

For parts (b),(c) and (d), use the trial state-vector in the coordinate representation given by the wave-function:

$$\psi(r,\theta,\phi) = \frac{1}{A}e^{-\beta r^2},$$

in order to estimate the ground-state energy of the hydrogen atom. Here r, θ, and ϕ are spherical coordinates, β is a variational parameter, and A is a normalization constant.

(b) What is the orbital angular momentum quantum number l of this Gaussian wave function? What can be the motivation for choosing this value of l for the trial wave function?

(c) Compute $E(\beta) = \langle\psi|H|\psi\rangle$ as a function of β. To simplify the calculation, use the result (normalization included here)

$$\langle\psi|T|\psi\rangle = \frac{3\hbar^2\beta}{2m}$$

for the expectation value of the kinetic energy of the electron T of the mass m, $\hbar$ is a reduced Planck's constant, and evaluate the expectation value of the potential energy $V = -\frac{e^2}{r}$.

Useful integrals:

$$\int_0^\infty e^{-\beta x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}, \quad \int_0^\infty re^{-\beta r^2}dr = \frac{1}{2\beta}, \quad \int_0^\infty r^2e^{-\beta r^2}dr = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}.$$

(d) Use your result in (c) to determine the best estimate for the ground-state energy of the hydrogen atom for a Gaussian trial wave function. Express your result in terms of e, m, and $\hbar$. Also, given the exact ground-state energy $E_0 = -\frac{me^4}{2\hbar^2}$, why your result is different?

B6. Modern Physics II

Rutherford's α-particle (kinetic energy $K_\alpha = 5$ MeV) scattering experiment revealed that $\sim$1 in 8000 particles scattered off gold-foil target ($Z = 79$) at angles $\theta > 90°$, contradicting the Thomson 'plum pudding' model, which predicted negligible large-angle scattering.

a) Describe the Thomson model and the rationale for its construction. Explain why this model cannot account for large-angle scattering events and why most the atom's mass should be concentrated in the positively charged nucleus.

b) The observation of 180° scattering implies the nucleus is smaller than the distance of closest approach b_c. Calculate this upper bound on nuclear radius. You can assume the nucleus remains immobile during collision. Why is this a reasonable assumption? How does this compare to the atomic radius?

c) Using the crude approximation that large-angle scattering occurs only when the impact parameter is less than the distance of closest approach b_c, estimate the cross-section as $\sigma = \pi b_c^2$. Given a gold-foil thickness of $\sim 1\mu$m and atomic spacing of ~ 3 Å, calculate the probability of large-angle scattering and verify it is comparable to the observed 1/8000 ratio.