Ph.D. QUALIFYING EXAMINATION – PART A

Tuesday, January 11, 2022, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed.

A1. An asteroid of mass m approaches the earth with impact parameter b and relative velocity v. Ignore the influence of the Sun and Moon on the asteroid and the Earth.

a) What is the initial angular momentum of the asteroid with respect to the Earth?

b) If the Earth were a point mass, what would the closest approach of the asteroid to the Earths center be? Express your answer in terms of b, v, R_e (the actual radius of the Earth) and g (the acceleration due to gravity on Earth's surface)

c) Will the asteroid hit the Earth? Take $b = 20R_e$ and $v = 1000\ m/s$.

A2. A relativistic K^0 meson (rest mass $E_{K^0} = 497.6MeV$) is traveling with a speed of $v_0 = 0.9c$ when it decays into a π^+ meson and a π^- meson (both with rest mass $E_{\pi^\pm} = 139.6MeV$. What are the greatest and the least speeds that the mesons may have?

A3. Consider two interacting particles. The first particle has spin-1 and the second particle has spin-1/2.

a) What are the possible values for the total spin angular momentum of the two-particle system? How many independent (spin) states does the system have?

b) Suppose the two particles interact via the Hamiltonian that is given by

$$H = \frac{a}{\hbar^2}\overrightarrow{S_1} \cdot \overrightarrow{S_2} + \frac{b}{\hbar}(S_{1z} + S_z)$$

Find all the eigenvalues of the Hamiltonian.

A4. A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field $\vec{E}_0 = E_0\hat{z}$. The sphere has a radius a and a dielectric constant $\kappa = \varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$.

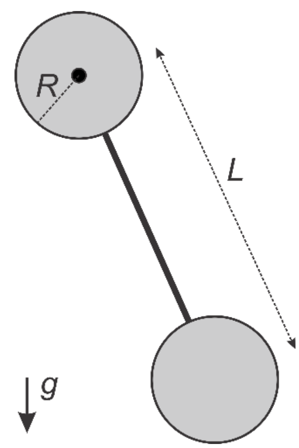
a) Determine the electric potential inside and outside the dielectric sphere.

b) Determine the electric field inside the dielectric sphere.

c) Determine the polarization $\vec{P}$ inside the dielectric sphere.

Recall: If there is azimuthal symmetry and the charge density, $\rho = 0$, the general solution to $\nabla^2 V = 0$ in spherical coordinates is: $V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$, where $P_\ell(\cos\theta)$ is the Legendre polynomial. Note: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$ and $\int_{-1}^{1} P_\ell(x)\, P_{\ell'}(x)dx = \frac{2\delta_{\ell\ell'}}{2\ell+1}$.

A5. **Two-disk pendulum**



Consider two uniform disks, each of mass *M* and radius *R*. They are rigidly connected by a massless rod such that their centers are a distance *L* apart. One of the disks (the upper one in the picture) is pivoted by a frictionless pin through its center.

a) Find the frequency of small oscillations the system performs under the influence of gravity.

b) How does the frequency change if the second disk (the lower one in the picture) is mounted to the rod by a frictionless bearing at its center so that it is free to spin?

A6. Consider a one-dimensional harmonic oscillator with mass m and electric charge e. At the time $t = 0$, a uniform electric field F is suddenly added to the system. Namely, the Hamiltonian of the system is given by

$$H = \begin{cases} \dfrac{p^2}{2m} + \dfrac{m\,\omega^2 x^2}{2} & (t < 0) \\ \dfrac{p^2}{2m} + \dfrac{m\,\omega^2 x^2}{2} - eFx & (t \geq 0). \end{cases}$$

You may use the fact that the Hermite polynomials $H_n(z)$ satisfy the following equations:

$$\left(\frac{d^2}{dz^2} - 2z\,\frac{d}{dz} + 2n\right) H_n(z) = 0.$$

$$e^{-s^2+2sz} = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(z).$$

$$\int_{-\infty}^{\infty} H_n(z) H_m(z) e^{-z^2} dz = \sqrt{\pi}\, 2^n n!\ \delta_{nm}.$$

(1) At $t < 0$, the solution of the stationary Schrödinger equation for an n-th energy eigenstate is given by

$$\psi_n(x) = H_n\left(\frac{x}{\lambda}\right) f(x)$$

$$\lambda = \sqrt{\frac{\hbar}{m\omega}}$$

$$\varepsilon_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad (n = 0, 1, 2, \dots)$$

Find the function $f(x)$ including a proper normalization factor.

(2) At $t \geq 0$, solve the stationary Schrödinger equation to find the energy eigenstates E_n and normalized eigenfunctions, $\varphi_n(x)$ $(n = 0,1,2 \ldots)$. You may use the result $\psi_n(x)$ from 1).

(3) Assume that the harmonic oscillator is in its ground state at $t < 0$. Calculate the probability that this state appears in the n-th eigenstate at $t \geq 0$. Also, discuss what kind of probabilistic distribution it follows.

Hint: At $t \geq 0$, the wavefunction in the time-dependent Hamiltonian is given by

$$\Psi(x,t) = \sum_{n=0}^{\infty} A_n \varphi_n(x) e^{-iE_n t/\hbar},$$

where A_n is an expansion coefficient with respect to the eigenfunctions $\varphi_n(x)$.

Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, January 12, 2022, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed.

B1. Consider a point particle of mass m subject to a gravitational force $-mg\hat{z}$ and constrained to move on the 2-dimensional frictionless surface of a lower hemisphere of radius R centered at the origin.

a) Write down the Lagrangian for the problem, and from it find the Lagrange-Euler equations for the motion of the particle in r, θ, ϕ spherical coordinates.

b) The particle is orbiting the hemisphere at a constant polar angle θ_0 with respect to the z-axis, where $\frac{\pi}{2} < \theta_0 < \pi$. At time $t = 0$, the particle is at $(r, \theta, \phi) = (R, \theta_0, 0)$ and its velocity is in the $+\hat{\phi}$ direction. Find the coordinates of the particle for $t > 0$.

c) Suppose the orbiting particle in part (b) is given an instantaneous impulse in the $-\hat{\theta}$ direction such that the resulting trajectory reaches the minimum polar angle θ_1, where $\frac{\pi}{2} < \theta_{min} < \theta_0$. Find the equation that describes the maximum polar angle θ_{max} that the particle can take in terms of known parameters of the problem ($m, R, \theta_0, \theta_{min}, g$, not all of them have to be involved). Do not attempt to solve the equation for θ_{max}.

d) In part (c), if you assume that $|\theta_{min} - \theta|$ is small, then you can linearize one of the Lagrange-Euler equations as a harmonic oscillator equation for θ around θ_0. Find the frequency of this oscillation.

B2. In a carbon monoxide molecule CO, the transition from the $\ell = 2$ to the $\ell = 1$ state results in the emission of a $\Delta E = 9.55 \times 10^{-4} eV$ photon. [$\hbar = 1.055 \times 10^{-34} J \cdot s$, proton mass $M_p = 1.67 \times 10^{-27} kg$, mass of C is 12amu, mass of O is 16amu]

(a) Find the moment of inertia of the molecule.

(b) What is the bond length of this molecule?

B3. The potential due to a given local charge distribution around the origin of the coordinate system is given by $V(\vec{r}) = k\int \frac{\rho(\vec{r'})}{|\vec{r}-\vec{r'}|} d^3r'$, where $k = \frac{1}{4\pi\varepsilon_0}$.

a) If the total charge is zero, show that the leading order term in the potential for large distances has the form $V(\vec{r}) = k\,\vec{r}\cdot\vec{p}/r^3$, where the electric dipole moment is given by $\vec{p} = \int d^3r'\, \overrightarrow{r'}\varrho\left(\overrightarrow{r'}\right)$.

b) Consider an additional point charge q located at a point $\vec{r}$ lying far from the dipole; the interaction energy is given by $U = k\vec{p}\cdot q\,\vec{r}/r^3$. Show that we can interpret this energy as the interaction energy of the dipole moment with the electric field $\vec{E}$ produced by q at the origin, i.e., $U = -\vec{p}\cdot\vec{E}$.

c) Use $U = -\overrightarrow{p_1}\cdot\vec{E}$ as the interaction energy of an electric dipole of moment $\overrightarrow{p_1}$ with the electric field $\vec{E}$ produced by a given charge distribution far from $\overrightarrow{p_1}$. Calculate the interaction energy U for a dipole-dipole interaction, i.e., for the interaction of $\overrightarrow{p_1}$ at the origin with the field $\vec{E}$ produced by another dipole moment $\overrightarrow{p_2}$ located at $\vec{r}$.

B4. A steady current flows in the z-direction down a long hollow cylindrical wire of inner radius a and outer radius b . The current density is distributed as $\vec{J}(s) = Ks\hat{z}$ for $a \le s \le b$ and equal zero otherwise. K is a constant and s is the cylindrical coordinate perpendicular to the z direction. The cylindrical coordinates used here are defined as follows:

$0 \le s < \infty,\ \ 0 \le \phi \le 2\pi,\ \ -\infty < z < \infty$.

a) Determine the magnetic field in all three regions: $0 \le s \le a,\ \ a \le s \le b,\ \ b \le s < \infty$.

b) Determine the vector potential, $\vec{A}$,in all three regions: $0 \le s \le a,\ \ a \le s \le b,\ \ b \le s < \infty$.

Hint: Use the fact that: $\vec{B} = \vec{\nabla}\times\vec{A}$ and in cylindrical coordinates

$$\vec{\nabla}\times\vec{A} = \left[\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial\left(sA_\phi\right)}{\partial s} - \frac{\partial A_s}{\partial\phi}\right]\hat{z}$$

B5. **Absorption of ideal gas**

Consider a classical ideal gas consisting of atoms of mass m. The gas is at temperature T and pressure p. When the gas is brought into contact with a solid surface, atoms are absorbed by the surface if they hit it with a velocity component normal to the surface of at least v_0. Atoms that hit the surface with normal velocity components less than v_0 are reflected elastically.

Derive an expression for the absorption rate W (absorbed atoms per time and area of the surface) as a function of the mass m, pressure p, temperature T, and v_0.

B6. Consider a particle with a spin of 1/2 (and mass of 1 for simplicity) in one dimension, and this particle follows the Hamiltonian,

$$H_0 = \frac{1}{2}\{p^2 + A(x)^2\} + \frac{\hbar}{2}\sigma_3 \frac{dA(x)}{dx}$$

where $p = \frac{\hbar}{i}\frac{d}{dx}$, and $A(x)$ is a smooth function and $|A(x)| \to \infty$ as $x \to \infty$. The σ_3 is one of the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

that satisfy $\sigma_a\sigma_b = \delta_{ab} + i\sum_{c=1}^{3}\varepsilon_{abc}\sigma_c$ where ε_{abc} is the Levi-Civita symbol in three dimensions.

1) When $A(x) = x$, derive the energy states of the system, using the creation and annihilation operators:

$$a = \frac{1}{\sqrt{2\hbar}}(p - ix)$$

$$a^\dagger = \frac{1}{\sqrt{2\hbar}}(p + ix)$$

Also, show that all the non-zero energy states are doubly degenerate.

2) For an arbitrary function $A(x)$, let us define the following operators:

$$Q_{\pm} \equiv \frac{1}{\sqrt{2}} \{Q_1 \pm iQ_2\}$$

$$Q_1 \equiv \frac{1}{2} \{\sigma_1 p + \sigma_2 A(x)\}$$

$$Q_2 \equiv \frac{1}{2} \{\sigma_2 p - \sigma_1 A(x)\}$$

2a) First show that $H_0 = 2\,{Q_1}^2 = 2\,{Q_2}^2$ and $[H_0,\ Q_{\pm}] = 0$.

2b) Then using the results of (2a), prove that the degeneracy of energy eigenstates shown in 1) holds for any $A(x)$. [Hint: Consider $Q_{\pm}\,|E,-\rangle$ where $|E,-\rangle$ denotes a state with energy E and lower spin.

3) In the case of $A(x) = x$, add the following interaction term V

$$V = \lambda\,(Q_+ + Q_-) = \lambda\sqrt{\hbar}\begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix}.$$

Using **degenerate perturbation theory**, calculate the energy spectrum of $H = H_0 + V$ up to the first order of λ.