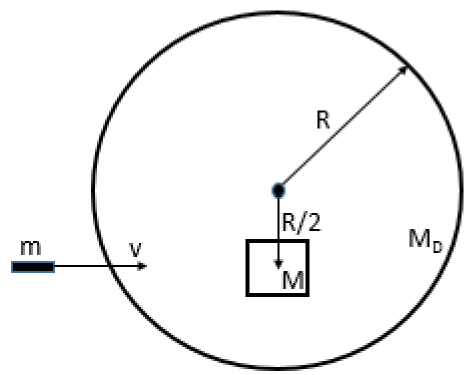
Ph.D. QUALIFYING EXAMINATION – PART A

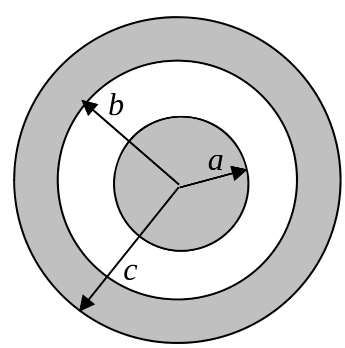
Tuesday, January 9, 2018, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outlines, 'Mathematical Handbook of Formulas and Tables'.

A1. A block of mass M is rigidly attached to a uniform disk of radius R at a distance of R/2 from the fixed axis of the disk. The disk-block system is free to rotate about the fixed axis. A bullet of mass m is fired with velocity v into the block and the bullet sticks in the block. The mass of the disk is M_D. Find the amplitude of the oscillations of the disk-block system after the collision.



A2. A non-conducting sphere of radius a carries a volume charge density $\rho = Ar^2$, where A is a constant. It is surrounded by a thick concentric metal spherical shell of inner radius b and outer radius c. The thick conducting shell carries no net charge.



a) Use Gauss's Law to determine the electric field as a function of r, *i.e.*, for $r < a$; $a < r < b$; $b < r < c$; $r > c$

b) Find the surface charge density σ at b and c.

c) Determine the electric potential as a function of r.

d) If the outer surface at $r = c$ is grounded, how do the results above change?

A3. A particle of mass m is confined by an infinite potential to the one-dimensional region $0 < x < L$. Initially, the particle is occupying the ground state of the potential. At $t = 0$ the potential well doubles in size to cover the region $0 < x < 2L$.

a) Write the expression for the normalized wave function of the particle for $t < 0$.

b) Calculate the probabilities that the particle transitions to (i) the ground, (ii) the first excited, and (iii) the second excited states of the expanded potential well after the change has taken place.

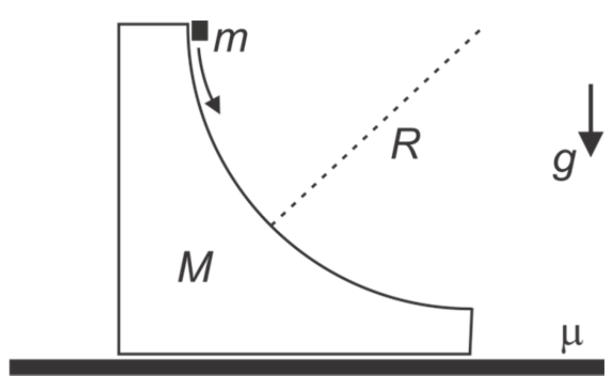
A4. A particle of mass m is in the ground state of a 1D square well potential confining it to the region $x \in [0, a]$. A heavy particle, moving with velocity v parallel to the x-axis, passes through the region at high speed and interacts with the particle through a weak short-range interaction that can be approximated by the time-dependent delta function potential $V(t) = \alpha\delta(x - vt)$, where $\alpha << \hbar^2/2ma$ is a small constant.

a) Quite generally, starting with the time-dependent Schrödinger equation, show that the state of a quantum system at time t, evolving under a Hamiltonian $H(t)$, can be expressed in terms of its state at an initial time t_0 through the relation

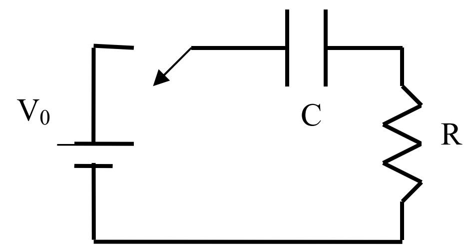
$$|\psi(t)\rangle = |\psi(t_0)\rangle - (i/\hbar) \int_{t_0}^{t} H(t')|\psi(t')\rangle dt'.$$

b) Use this last result, with $t_0 = -\infty$ and $t = +\infty$ to find, correct to lowest non-vanishing order α, the total probability for the particle to make a transition to the first excited state of the system as a result of this collision.

A5. A large block of mass M is at rest on a rough horizontal surface. The block contains a quarter-circle frictionless track of radius R as shown in the picture. A small mass m is released from rest at the top of the track. Find the minimum value of the static friction coefficient μ between the large block and the horizontal surface such that the large block remains at rest while the small mass is sliding down.



A6. A capacitor, C, is connected to a resistor, R, and to a battery, V_0, as shown. Start the time as $t = 0$ when the switch is thrown.



a) After the switch is thrown, determine the charge and current in the circuit as a function of time.

Assume the capacitor is a parallel plate capacitor constructed of circular plates of radius a and separated by a distance $w \ll a$. The wires connect to the centers of the plates.

b) Find the electric and magnetic fields between the plates, as a function of ρ and t.

c) Use the Poynting vector to determine the electromagnetic power flowing into the capacitor.

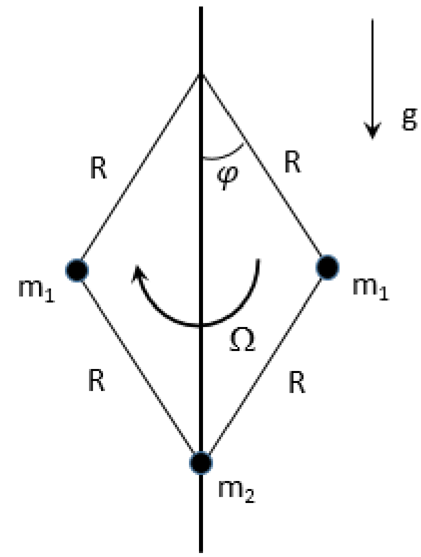
d) Determine the total energy stored in the capacitor after a long time, i.e., let $t \to \infty$.

Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, January 10, 2018, 1:00 – 5:00 p.m.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outlines', 'Mathematical Handbook of Formulas and Tables'.

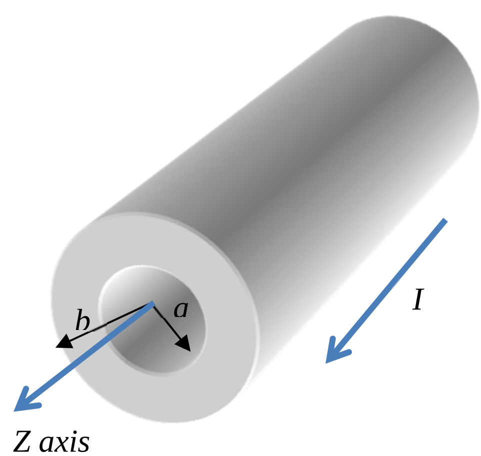
B1. Flyball Governor: In the system shown in the Figure, there are hinges at all the corners of the rhombus, and the top corner is attached to the vertical axis. The particle m_2 moves without friction on the vertical axis and the whole system rotates about this vertical axis with a constant angular velocity Ω. Gravity pulls down on all masses. Derive the Lagrangian of the system, obtain the equations of motion, and find the equilibrium angle φ_0.



B2. A steady current I flows down a long hollow cylindrical wire of inner radius a and outer radius b, where $b = 2a$. The current is uniformly distributed in the wire.

a) Use Ampere's Law to determine the magnetic field $\vec{B}$ in all three regions: $\rho < a, \quad a < \rho < b, \quad b < \rho$.

b) Determine the vector potential $\vec{A}$ in all three regions: $\rho < a, \quad a < \rho < b, \quad b < \rho$.



As you may know, in cylindrical coordinates

$$(\rho, \phi, z), \quad \vec{\nabla} \times \vec{A} = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \hat{z}\frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)$$

B3. Recall the twin paradox experiment. Twins Mary and Frank undertake an experiment in which Mary decides to travel with speed v to a star system (distance L away) and back. When, according to Frank's calculations, Mary is turning back home, Frank decides to rush to meet her on the way home. He computes his speed u in such a way that both Frank and Mary will be exactly the same age at the time they meet each other.

(a) Write an equation, which Frank used to compute the required speed u from Mary's speed v.

(b) Show that there is a maximum speed v_{max} when Frank still can accomplish his feat. Find the value of v_{max}.

Note: you can neglect the brief periods of acceleration/deceleration by both travelers during their journeys.

B4 The Hamiltonian for a particular two-state system is given by

$H = \begin{pmatrix} E_1^0 & \lambda A \\ \lambda A & E_2^0 \end{pmatrix}$. The energy eigenkets for $\lambda = 0$ are $\phi_1^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\phi_2^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(a) Solve the problem exactly $(\lambda \neq 0)$ to find the energy eigenkets (ψ_1 and ψ_2) and the corresponding eigenvalues (E_1 and E_2).

(b) Assume that $\lambda A \ll \left|E_1^0 - E_2^0\right|$ and solve the same problem using time-independent perturbation theory up to first order in the energy eigenkets and up to second order in the energy eigenvalues. Compare your results with those in part (a).

(c) Suppose the two unperturbed energies are almost degenerate, *i.e.*, $\lambda A \gg \left|E_1^0 - E_2^0\right|$. Show that the exact results obtained in part (a) closely resemble what you expect by applying degenerate perturbation theory to this problem with $E_1^0 = E_2^0$.

B5. A particle constrained to move in the xy-plane is subject to an isotropic linear restoring force directed towards the origin. The Hamiltonian is

$$H^{(0)} = \sum_{\alpha=x,y} \left(\frac{1}{2m} P_\alpha^2 + \frac{1}{2} m\omega^2 x_\alpha^2 \right) = \sum_{\alpha=x,y} \frac{\hbar\omega}{2} (p_\alpha^2 + q_\alpha^2)$$

where $q_\alpha = (m\omega/\hbar)^{1/2} x_\alpha$ and $p_\alpha = (m\hbar\omega)^{-1/2} P_\alpha$. [Recall that, for a 1D oscillator, the ground state energy eigenfunction can be written $\phi_0(q) = \langle q|\phi_0\rangle = \pi^{-1/4} e^{-(1/2)q^2}$.]

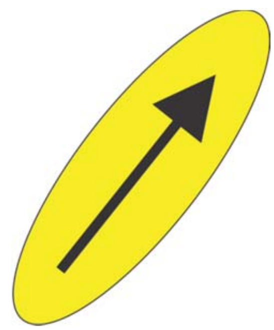
a. Using, e.g., the raising operators $a_\alpha^+ = 2^{-1/2}(q_\alpha - ip_\alpha)$, construct a set of orthonormalized wave functions $\psi_{1,0}$ and $\psi_{0,1}$ for the two energy eigenstates with $n = n_x + n_y = 1$.

b. Suppose this system is now subject to a weak, but strongly-localized perturbation

$$H^{(1)} = \lambda\hbar\omega\, \delta(q_x - q^0)\delta(q_y - q^0).$$

where $1 >> \lambda > 0$. Find, to first order in λ, the new energy eigenvalues associated with the $n = 1$ level for this system.

B6. N non-interacting spheroid-shaped magnetic nanoparticles are suspended in a liquid of volume V. Each nanoparticle has a permanent magnetic moment μ pointing along the symmetry axis of the spheroid (see figure). The classical Hamiltonian describing the rotation of a single nanoparticle in a magnetic field **B** pointing in the z-direction is given by



$$H_{\mathrm{rot}} = 1/(2I)\, (p_\vartheta^2 + p_\phi^2/\sin^2\vartheta) - \mu\, B \cos\vartheta\, .$$

Here, ϑ and ϕ are the usual angles of a spherical coordinate system, p_ϑ and p_ϕ are the canonically conjugate momenta, and I is the moment of inertia of the nanoparticle (for axes perpendicular to its symmetry axis).

(a) Calculate the rotational part Z_{rot} of the canonical partition and find the average magnetization per volume V of the liquid, $M = (N/V)\, \mu\, \langle\cos\vartheta\rangle$, as a function of field B and temperature T.
[Hint: $\int_{-\infty}^{\infty} dx\, \exp[-x^2/(2a^2)] = (2\pi a^2)^{1/2}$.]

(b) In the weak-field limit $\mu B << k_B T$, the magnetization behaves as $M = \chi\, B$ where χ is the magnetic susceptibility. Determine the magnetic susceptibility by expanding your result for M from part (a) in the limit $\mu B << k_B T$ to lowest non-trivial order.
[Hint: $\coth x = 1/x + x/3 + \ldots$]