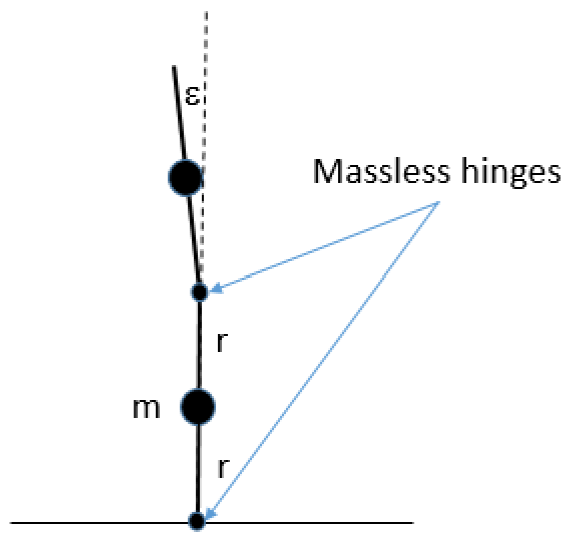
Ph.D. QUALIFYING EXAMINATION – PART A

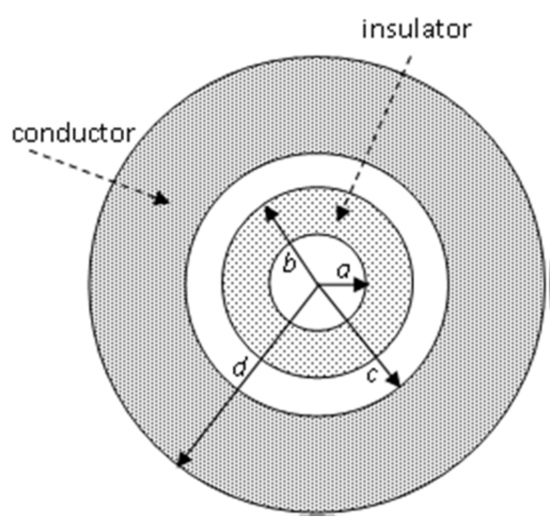
Tuesday, January 10, 2017, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outlines, 'Mathematical Handbook of Formulas and Tables'.

A1. 2. Two massless sticks of length 2r with mass m fixed in the middle are hinged at an end. One stands on top of the other as shown in the figure. The bottom end of the lower stick is hinged on the ground. They are held such that lower stick is vertical, while the upper stick is tilted by a small angle ε with respect to the vertical. They are then released. Find the initial angular accelerations of the sticks at the moment of time t=0. (Hint: consider the small angle approximation even **before** writing the kinetic energy for your Lagrangian)



A2. A solid insulating thick spherical shell of inner radius a and outer radius b has a uniform charge per unit volume $\rho(r) = A/r$ for $a < r < b$. Concentric with this spherical shell is an uncharged conducting spherical shell with inner radius c and outer radius d. There also is no charge in the regions $0 < r < a$ and $b < r < c$. The figure shows a cross section.



(a) Determine the electric field in all five regions:
$0 < r < a, \quad a < r < b, \quad b < r < c, \quad c < r < d, \quad r > d$
Give a rough sketch of $E(r)$ vs. r.

(b) Determine the electric potential in all five regions.
Give a rough sketch of $V(r)$ vs. r.

(c) How does $V(r)$ change if the conductor is grounded at $r = d$.

A3. Consider a diatomic molecule of mass $2M$ and separation distance a.

(a) Derive an order of magnitude estimate for the energy ΔE_{rot} of transitions between the rotational levels of the molecule.

(b) Derive an order of magnitude estimate for the energy ΔE_{vib} of transition between the vibrational levels of the molecule.

(c) How does the ratio between ΔE_{rot} and ΔE_{vib} depend on system parameters?

(d) Estimate the numerical values of ΔE_{rot} and ΔE_{vib} for a hydrogen molecule. Express your answer in electron volts.

A4. Consider, in three dimensions, an isotropic quantum mechanical harmonic oscillator, corresponding to the classical Hamiltonian

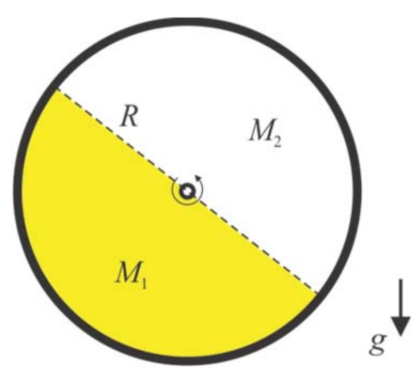
$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$$

Because the potential is spherically symmetric, the eigenstates of H can be chosen to also be eigenstates of L^2 and L_z where $\vec{L} = \hbar\vec{\ell}$ is the orbital angular momentum. After determining the appropriate normalization constant, obtain an optimized variational estimate of the energy levels of this system using a trial wave function of the form

$$\phi_\ell^m(r,\theta,\phi) = A\, e^{-r/a}\ Y_\ell^m(r,\theta,\phi)$$

where $Y_\ell^m(r,\theta,\phi)$ denotes the spherical harmonic with angular momentum quantum numbers ℓ and m. It may help to recall that the quantity $p^2 = (\vec{p}\times\hat{r})^2 + (\vec{p}\cdot\hat{r})^2 = L^2/r^2 + {p_r}^2$, can be resolved into tangential and radial contributions.]

A5. A sphere of radius R consists of two solid hemispheres of different masses M_1 and M_2 with $M_1 > M_2$ (the density within each hemisphere is uniform). The sphere can rotate about a horizontal axel that goes through its center and is in the plane separating the hemispheres as shown in the figure.



a) Find the location of the center of mass of a single hemisphere.

b) Compute the moment of inertia of a single hemisphere about the given axle.

c) Describe the stable mechanical equilibrium of the system.

d) Determine the frequency of small oscillations about this equilibrium.

e) Discuss the limit $M_1 \to M_2$.

A6. Fourier's law, $\vec{q} = -k\nabla T$, is exactly analogous to Ohm's Law, $\vec{J} = \sigma\vec{E}$; with $\vec{E} = -\nabla V$. In Fourier's law, $\vec{q}$ is heat flux, just like $\vec{J}$ is charge flux, *k* is the thermal conductivity, and *T* is the temperature. In fact, Ohm was inspired by Fourier's work on heat conduction.

An exactly analogous equation to Gauss's Law, $\nabla \cdot E = \rho / \varepsilon_o$, arises in Geophysics if one assumes the Earth is heated by point sources of uniform mass density ρ that output *H* watts/kg of radioactive heat, namely $\nabla \cdot \vec{q} = \rho H$, or

$$k\nabla^2 T + \rho H = 0$$

a) Assume spherical symmetry (recall $\nabla^2 T = (1/r)d^2/dr^2(rT)$) and uniform ρ, *H*, and *k*, to solve for *T(r)*. You're welcome to solve the electrostatics problem for *V* and translate variables afterwards.

b) A fundamental problem arises near the surface of the Earth for this model, in which ***all heat transport is by conduction***. Linearize the temperature profile by writing $r = a - y$, $y << a$, where *a* is the radius of the Earth, and show that

$$T(y) = T_0 + \frac{\rho H a}{3k} y$$

c) It turns out that the melting curve of forsterite, the hardest of the minerals to melt in the mantle of the Earth, is given by T_{melt} = 1898 + 1.54·(y/km) in °C.

Write *T(y)* above for *y* in *km*, assuming T_0 = 0 °C, $\rho = 3.3\cdot10^3$ kg/m^3, $H = 7.4\cdot10^{-12}$ W/kg,

$a = 6.4\cdot10^6$ m, and $k = 3$ W/K/m, and show that the two lines cross at about $y = 120$ km.

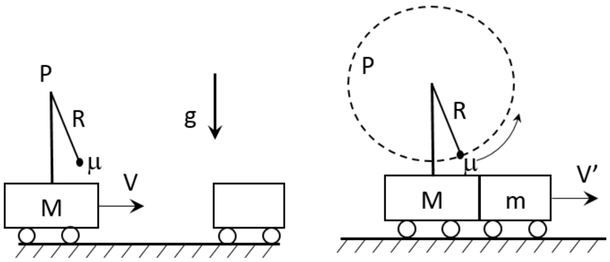
The mantle extends to $y = 3{,}000$ km! Thus, all the mantle of the Earth should be molten, which flatly contradicts the existence of seismic shear waves in the mantle, which mean the mantle is solid. This is the reason we know the mantle must convect, a far more efficient mode of heat transport. The mantle convects on long time-scales because it is a solid near the melting point; exactly the reason ice flows in a glacier.

Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, January 11, 2017, 1:00 – 5:00 p.m.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outlines', 'Mathematical Handbook of Formulas and Tables'.

B1. A cart of mass M has a pole on it from which a ball of mass μ hangs from a thin string attached at point P. Assume the ball hangs straight down initially. The cart and ball have initial velocity V. The cart crashes into another cart of mass m and sticks to it (see Figure). If the length of the string is R, show that the smallest initial velocity for which the ball can go in a circle around point P is $V = \frac{m+M}{m}\sqrt{5gR}$. Neglect friction and assume $\mu \ll M, m$.



B2. A surface charge density $\sigma(\phi) = \sigma_1 \cos(2\phi)$ is glued over the surface of cylinder of radius *R*. The cylinder is "infinitely long" in the *z* direction. Note: σ_1 is a constant.

a) Determine the electric potential inside and outside the cylinder.

b) Suppose $\sigma(\phi) = \sigma_0 + \sigma_1 \cos(2\phi)$, where σ_0 is a constant surface charge density. How does your answer change? Hint: use the principle of superposition to break the problem into two simpler problems. That is, solve for the potential due to σ_0 by finding *E* and then finding *V* and adding your result for σ_0 to your solution in part (a).

Solution to Laplace's equation, $\nabla^2 V = 0$, in cylindrical coordinates (ρ, ϕ no *z* dependence):

$$V(\rho,\phi) = A_0 + B_0 \ln \rho + \sum_{m=1}^{\infty} \rho^m (A_m \cos m\phi + B_m \sin m\phi) + \sum_{m=1}^{\infty} \frac{1}{\rho^m}(C_m \cos m\phi + D_m \sin m\phi)$$

B3. According to a fixed *xy*-coordinate system a photon is moving at an angle ϑ with respect to the **positive** *x*-axis so that the velocity of the photon is given by $\vec{u} = (u_x, u_y) = (c\cos\theta, c\sin\theta)$, where c is the speed of light. A plane mirror lying in the *yz*-plane is moving with a speed v in the **negative** *x*-direction with respect to the fixed *xy*-coordinate system. The photon collides with the mirror and is reflected.

(a) Using the relativistic velocity transformation, derive expressions for the two components of the velocity of the photon with respect to the moving mirror's frame of reference after reflection.

(b) Find the new direction of the photon [i.e. the angle ϑ' it makes with the **negative** *x*-axis] in the fixed reference frame. Verify explicitly that $\vartheta' = \vartheta$ in the limit of $v \to 0$.

(c) Verify that the speed of the photon in the fixed reference frame after reflection is still equal to the speed of light c.

B4. Consider a quantum mechanical system for which the time-independent Hamiltonian H can be written $H = \varepsilon_0 K$, where ε_0 is a positive real energy (i.e., a scalar), and K is an operator that, in addition to being obviously **Hermitian**, also happens to be **unitary**.

a) Is H unitary also? Explain why it is, or why it is not.

b) For any positive integer n, reduce the operators K^{2n} and K^{2n+1} to their simplest form, expressing your answer in terms of single powers of K and/or other well-known operators and scalars. Use these results to express the evolution operator $U(t) = e^{-iHt/\hbar}$ as a polynomial of finite degree in H.

c) Suppose $|\varepsilon\rangle$ is an eigenstate of H with eigenvalue ε. Determine the spectrum of H, i.e., set $\{\varepsilon\}$ of possible energy eigenvalues.

e) Define two related operators $P_+ = \frac{1}{2}(1+K)$ and $P_- = \frac{1}{2}(1-K)$. Show that $P_+P_- = 0$, and that $P_\pm^n = P_\pm$ for any integer $n > 1$.

f) Show that for any state $|\psi\rangle$, the state $|\psi_\pm\rangle = P_\pm|\psi\rangle$ obtained by applying the operator $P_\pm$ to $|\psi\rangle$, if it does not vanish, is an eigenstate of H and determine the associated eigenvalue. Under what conditions will $P_\pm|\psi\rangle$ vanish?

B5. Consider a 1-dimensional attractive delta function potential, $V(x) = -\alpha\,\delta(x)$.

a) Calculate the bound state energy for a particle of mass m in the presence of this attractive delta function potential.

b) Determine the normalized wave function for the bound state energy in part (a).

c) Consider a particle of mass m and positive energy E incident on the attractive delta function potential given above. Calculate the transmission and reflection coefficients for this particle.

B6. A vessel of volume V contains N_A particles of type A and N_B particles of type B. The A particles are small and can be treated as an ideal gas of point particles of mass m at temperature T. The B particles are spherical with radius R. They are much heavier than the A particles and can be treated as stationary but randomly distributed throughout the volume.

Find the collision rate between the A and B particles, i.e., how many collisions between A and B particles occur per time in the vessel?