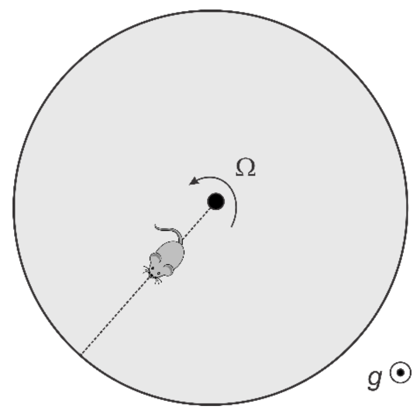
Ph.D. QUALIFYING EXAMINATION – PART A

Tuesday, August 22, 2023, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outline, 'Mathematical Handbook of Formulas and Tables'.

A1. **Mouse on a disk**



A mouse of mass M is sitting at the center of a large horizontal disk that rotates about its center with constant angular velocity Ω. At time $t = 0$, the mouse starts moving at a constant speed, v_0 , along a radial line on the disk (see top view to the right). The system is under the influence of a constant gravitational acceleration g, and the coefficient of static friction between the mouse and the disk is μ.

a) Find the acceleration of the mouse as a function of time in polar coordinates. Draw a diagram showing the acceleration vector at some time $t > 0$.

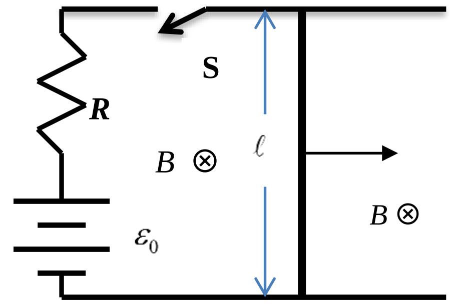
b) Find the time at which the mouse starts to slide on the disk.

c) Find the angle of the friction force with respect to the instantaneous position vector **r** just before the mouse starts to slide.

A2. A particle with rest-mass M, traveling at a relativistic speed v_0 along the x-axis, decays into two identical particles with masses m. After the decay, the newly created particles are observed to travel symmetrically with respect to the x-axis.

Find the angle between the direction of each particle and the x-axis.

A3. Consider a simplistic "rail gun" which consists of a metal bar of mass m able to slide frictionlessly on two parallel conducting rails a distance ℓ apart. A resistor R and a battery of emf ε_0 is connected across the rails and a uniform magnetic field $\vec{B}$, pointing into the paper, fills the entire region. The bar is initially at rest.



a) When the switch is initially closed, what will be the initial current I_0 in the circuit?

b) As the bar moves there will be a back emf generated. Determine the back emf generated if the bar has a speed v.

c) The back emf will oppose the emf of the battery. Determine an expression for the current now that the bar is moving with speed v.

d) Determine the magnetic force on the bar.

e) Determine the speed of the bar as a function of time assuming the bar starts from rest.

f) Does the bar have a terminal velocity? If so, what is it?

A4. The spin states of two electrons ($s_{1,2} = 1/2$) interact via the term $\hat{H} = \alpha\, \widehat{\vec{s_1}} \cdot \widehat{\vec{s_2}}$, where α is a constant. Ignore the spatial (electrostatic) interaction altogether. At time $t = 0$ the first electron is in the spin-up state relative to some axis, while the other one is in the spin-down state.

a) Show that eigenstates of $\widehat{s^2}$ and $\hat{s}_z$ where $\hat{\vec{s}} = \hat{\vec{s}}_1 + \hat{\vec{s}}_2$ and $\hat{s}_z = \hat{s}_{1z} + \hat{s}_{2z}$ diagonalize the Hamiltonian of the system.

b) Express the initial state of the system at $t = 0$ in terms of the eigenstates of $\widehat{s^2}$ and $\hat{s}_z$

c) Find the subsequent state of the system as a function of time using the appropriate basis.

d) What is the probability of finding the system in the initial (t = 0) spin-up/spin-down state at an arbitrary time t?

You might (or might not) find the following useful: $\hat{l}^{\pm}|lm\rangle = \sqrt{l(l+1) - m(m \pm 1)}\,|l, m \pm 1\rangle$

A5. **Quantum Mechanics**

A two-level atom coupled with an optical cavity can be described in the Jaynes-Cummings model, with a total Hamiltonian of

$$\hat{H}_{\mathrm{JC}} = \hat{H}_{\mathrm{o}} + \hat{H}_{\mathrm{a}} + \hat{H}_{\mathrm{int}} .$$

Here, $\hat{H}_{\mathrm{o}} = \hbar\omega_o(\hat{n} + \frac{1}{2})$ is the Hamiltonian of the cavity with eigenstates $|n\rangle$ (with $\hat{n}|n\rangle = n|n\rangle$, and $\hat{n} = \hat{a}^\dagger\hat{a}$ can be expressed in terms of the creation and annihilation operators).

The term $\hat{H}_{\mathrm{a}} = \hbar\omega_a|e\rangle\langle e|$ is the Hamiltonian of the atom with the ground state $|g\rangle$ of zero energy and the excited state $|e\rangle$ of energy $\hbar\omega_a$.

The interaction between the oscillator and the atom is given by:

$$\hat{H}_{\mathrm{int}} = \hbar k\hat{a}\hat{\sigma}^+ + \hbar k\hat{a}^\dagger\hat{\sigma}^-,$$

where $\hat{\sigma}^+ = |e\rangle\langle g|$ and $\hat{\sigma}^- = |g\rangle\langle e|$ being the raising and lowering operators between the ground and excited atomic states.

a) Show that $|0, g\rangle = |0\rangle|g\rangle$ is an eigenstate of the coupled system and calculate its energy.

b) To find all eigenstates of $\hat{H}_{\mathrm{JC}}$, we define the operator:

$$\hat{N} = \hat{n} + |e\rangle\langle e|,$$

which corresponds to the total number of excitations in the system. Show, that $\hat{N}$ is a conserved quantity (i.e., the eigenvalues of $\hat{N}$ are good quantum numbers).

c) Find all eigenstates of $\hat{N}$ and identify the degree of degeneracy associated with each eigenvalue of $\hat{N}$.

Useful relations: $\hat{a}|n\rangle = \begin{cases} \sqrt{n}|n-1\rangle & n > 0 \\ 0 & n = 0 \end{cases};$ $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

A6. **2D Motion of a Rigid Body**

As shown in the figure below, consider the two-dimensional motion of two point particles, each with mass m. They are connected to a massless rigid bar of length L. The center of mass position of this this system O_G is specified by a polar coordinate (R, θ), and the angular direction of the bar is defined by ϕ, as shown. The system is influenced by a central force whose potential energy is given by $U(x, y) = -\frac{GM}{\sqrt{x^2+y^2}}$.

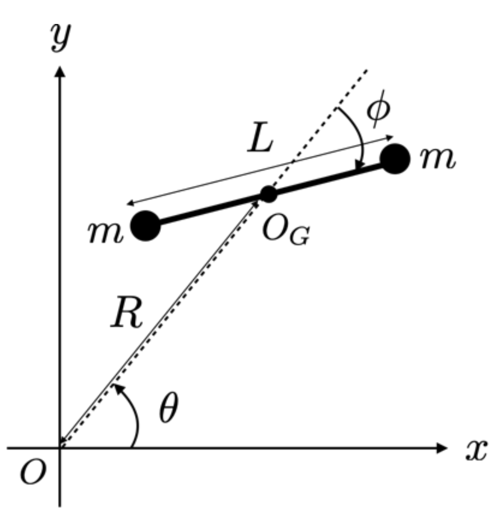
a) Write down the kinetic energy of this system, $T(R, \theta, \phi)$, and then show that it is a sum of two terms: translational and rotational kinetic energies.

b) Taking the limit of $L/R \ll 1$, find the Lagrangian of the system up to second order in L/R. Write down the equations of motion for the three variables. You may use the following expansion formula.

$$\frac{1}{\sqrt{1 + ax + bx^2}} \cong 1 - \frac{a}{2}x + \frac{(3a^2 - 4b)}{8}x^2$$

c) In the limit of $L/R \ll 1$, ϕ has an infinitesimal oscillatory solution around a stable point. Using the Lagrange equations of motion in b), find the angular frequency of the oscillation.

Hint: At $L = 0$, the system follows a circular motion with $\dot{\theta} = \omega_0$ (constant) and $R = R_0 = (6M/{\omega_0}^2)^{1/3}$ (constant), as expected. Find the oscillatory solution around the circular motion solution.



Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, August 23, 2023, 1:00 – 5:00 p.m.

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B1. **Simple Hamiltonian Dynamics**

Consider a one-dimensional harmonic oscillator of a point mass m with an angular frequency ω whose Lagrangian is given by

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

a) Find a Hamiltonian of the system, $H(x, p)$.

Consider the following canonical transformation from $(x, p = m\dot{x})$ to (Q, P) such that one of the phase-space coordinates becomes cyclic in the harmonic oscillator:

$$x = f(P)\sin Q$$

$$p = m\omega f(P)\cos Q$$

b) Determine the function $f(P)$ in terms of the given symbols.

c) For a generating function $F = F(x, Q)$, show that that the following equations must be satisfied for the transformation to be canonical. Also, find the generating function $F = F(x, Q)$ using these equations.

$$p = \frac{\partial F}{\partial x},$$

$$P = -\frac{\partial F}{\partial Q}.$$

d) Solve Hamilton's equations of motion in (Q, P) with $Q(t = 0) = 0$ and the total energy E. Also, plot the motion of this system in the phase space, (Q, P).

B2. A charged particle with charge e is restricted to move on a thin spherical shell of radius a and is placed in a weak uniform electric field E.

a) Identify eigenstates and eigenfunctions of the unperturbed problem (E=0). (If you have to compute anything here, you can, but you do not have to show your work here if you just know what they are.)

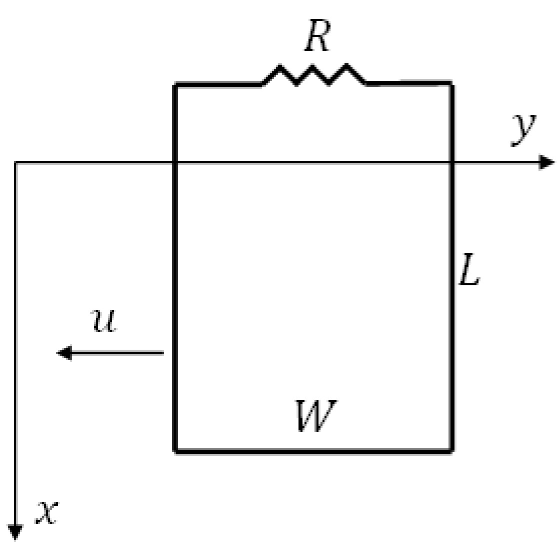
b) Use perturbation theory up to second order to calculate the energy splitting due to the electric field.

c) In part b), why can you use non-degenerate perturbation theory, even though the spectrum (hopefully) has degeneracies?

Useful formulas: $E_i = E_{i0} + \langle i|V|i\rangle + \sum_{i\neq j} \frac{|\langle i|V|j\rangle|^2}{E_{i0}-E_{j0}} + \cdots$

$$cos\theta\, Y_l^m(\theta,\phi) = \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}\, Y_{l+1}^m + \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}\, Y_{l-1}^m$$

B3. Consider a loop of wire, depicted in the diagram, of length L and width W. The loop is moving to the left at a constant speed u. At time $t_0 = 0$, the left edge of the loop is at $y = 0$. An external magnetic field of $B_0 e^{-\alpha y}\hat{\mathbf{z}}$, directed along z-axis exists throughout the domain of the problem. Resistance of the loop is R.



a) Find the time-dependent current induced in the loop.

b) What is the direction of the current?

B4. **Modern**

Muons are created in the upper atmosphere by cosmic radiation with an average speed of $0.994c$. At rest, a muon has a lifetime of about $2.16\ \mu$s.

a) How far (for an observer resting on earth) does an (average) muon travel during its lifetime? ($c = 2.998 \cdot 10^8\ \frac{\text{m}}{\text{s}}$)

b) What is the Feynman-diagram for muon decay?

B5. A sphere of radius R carries a volume charge density

$$\rho(r) = Ar^2 \text{ for } 0 \le r \le R \text{ (}A\text{ is a constant).}$$

a) Determine the electric field, $\vec{E}(r)$, inside and outside the sphere.
b) Determine the electric potential, $V(r)$, inside and outside the sphere.
c) Determine the total energy of the configuration.
d) Determine the net force that the southern hemisphere exerts on the northern hemisphere.

B6. **Two-dimensional ideal gas**

Molecules moving on the surface of a crystal can be approximately treated as a two-dimensional classical ideal gas. Consider such a gas consisting of N molecules of mass m on a square surface of area A in thermal equilibrium at temperature T.

a) Write down the Maxwell-Boltzmann velocity distribution for this two-dimensional ideal gas.
b) Find the average speed $<v>$ and the mean-square velocity $<v^2>$ in terms of m and T.
c) A small opening of length L is made in the boundary of the area. Find the effusion rate, i.e., the rate of particles escaping per unit of time.
d) Find the average energy of an effusing particle and compare it with the average energy of a particle in the gas.

$\int_0^\infty dx\, x\, exp(-ax^2) = 1/(2a)$, $\int_0^\infty dx\, x^2\, exp(-ax^2) = \sqrt{\pi}/\left(4a^{3/2}\right)$,

$\int_0^\infty dx\, x^3\, exp(-ax^2) = 1/(2a^2)$, $\int_0^\infty dx\, x^4\, exp(-ax^2) = 3\sqrt{\pi}/\left(8a^{5/2}\right)$,

$\int_0^\infty dx\, x^5\, exp(-ax^2) = 1/a^3$