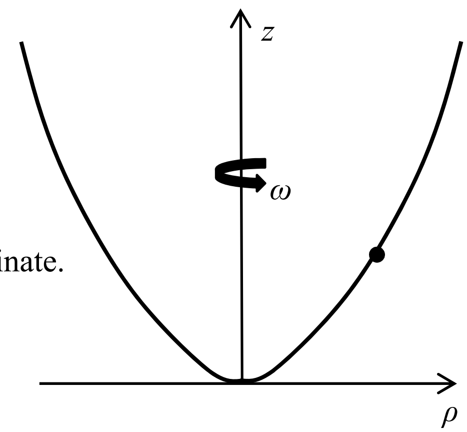
Ph.D. QUALIFYING EXAMINATION – PART A

Tuesday, January 3, 2012, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outlines, 'Mathematical Handbook of Formulas and Tables'.



A1. A bead of mass M slides without friction on a wire bent in the shape of a parabola that is being rotated with constant angular velocity ω about its vertical axis. Use cylindrical coordinates and let the equation of the parabola be $z = k\rho^2$.

a) Write down the Lagrangian in terms of ρ as the generalized coordinate.

b) Find the equation of motion of the bead and determine if there are positions of equilibrium.

c) Determine the stability of all equilibrium positions you find.

A2. Develop a quantitative model of linear thermal expansion. Assume that atoms in a solid are confined in one direction by a one-dimensional potential energy function that for small displacements goes as $V(x) = ax^2 - bx^3$, where a and b are constants. Assume b is small and using the classical Maxwell-Boltzmann distribution function, $e^{-V/kT}$, derive an expression for $\langle x \rangle$. Show that it is linearly proportional to the temperature and the anharmonicity coefficient b.

A3. The atoms in the atmosphere of a star emit light. The emission frequency of a particular element is f_0 if the atom is at rest. Due to the thermal motion of the atom the observed frequency is approximately shifted (via the Doppler effect) to $f = f_0\left[1-(v/c)\cos\Theta\right]$, where v is the speed of the atom and Θ is the angle between the directions of its motion and the observation. The atoms can be treated as a classical non-relativistic ideal gas at temperature T.

a) Calculate the resulting width $\Delta f = \sqrt{\langle f^2 \rangle - \langle f \rangle^2}$ of the spectral line observed on earth. (Here, $\langle \ldots \rangle$ denotes the average over all atoms of this particular element in the atmosphere.)

b) Determine the shape of the spectral line, i.e., the probability distribution $P(f)$.

A4. Let φ_1 and φ_2 be two normalized eigenfunctions of the nondegenerate Hamiltonian H_0, and let $\varepsilon_1 = \varepsilon_0 - \frac{1}{2}\Delta$ and $\varepsilon_2 = \varepsilon_0 + \frac{1}{2}\Delta$ be the corresponding energies. For this same system let Q be an observable with two eigenfunctions

$$\varphi_q = \frac{1}{\sqrt{2}}[\varphi_1 + \varphi_2] \text{ and } \varphi_{-q} = \frac{1}{\sqrt{2}}[\varphi_1 - \varphi_2]$$

having eigenvalues q and $-q$, respectively.

a) If at $t = 0$ the system is in the initial state $\psi(0) = \varphi_{-q}$what will the state of the system be at an arbitrary time > 0 ?

b) Evaluate the following matrix elements
$< \varphi_q|H_0|\varphi_q >, < \varphi_q|H_0|\varphi_{-q} >, < \varphi_{-q}|H_0|\varphi_q >, < \varphi_{-q}|H_0|\varphi_{-q} >$.

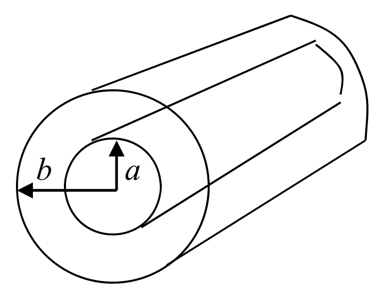
c) Evaluate the following matrix elements
$< \varphi_1|Q|\varphi_1 >, < \varphi_1|Q|\varphi_2 >, < \varphi_2|Q|\varphi_1 >, < \varphi_2|Q|\varphi_2 >$.

d) Find as a function of λ those energy eigenvalues of the perturbed Hamiltonian

$$H = H_0 + \lambda Q$$

which reduce to $\varepsilon_1$and ε_2 as λ goes to zero.

A5. A long coaxial cable carries a uniform volume charge density ρ on the inner cylinder (radius a), and a uniform surface charge density on the outer cylindrical shell (radius b). This surface charge is negative and of just the right magnitude so that the cable as a whole is electrically neutral. Assume cylindrical coordinates (s, ϕ, z) since ρ is the volume charge density.

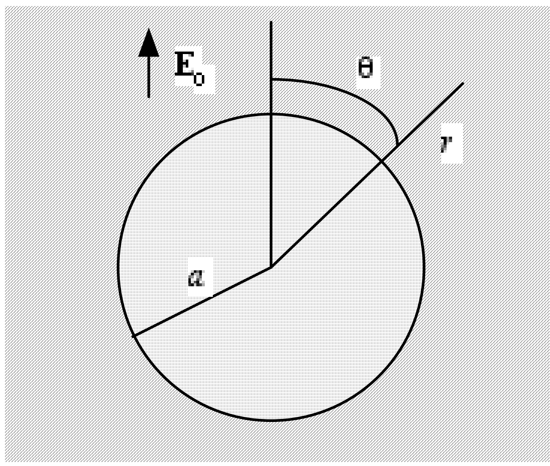


a) Use Gauss's Law to determine the electric field in each of the three regions: inside the inner cylinder $(s < a)$, between the cylinders $(a < s < b)$, outside the cable $(s > b)$.

b) Determine the electrostatic potential in each of the three regions listed above.

c) Determine the energy per unit length stored in the cable.

A6. A conducting sphere of radius a is covered by a thin insulating coating (a dielectric shell of negligible thickness) and placed in an infinite conducting medium. A uniform external electric field $\mathbf{E_0}$ is then applied, as shown in the figure. The external conducting medium is ohmic, therefore the current density $\mathbf{J} = \sigma\mathbf{E}$ everywhere, where σ is the conductivity of the conducting medium.



a) Since charges can only accumulate on the inner and outer surface of the insulator, sketch the distribution of charge once equilibrium (for the insulated spherical conductor) and a steady state current (for the external conducting medium) is reached. What is the value of the electric field inside the conducting sphere? What is the value of the electric field far from the conducting sphere?

b) Use the continuity equation obeyed by $\mathbf{J}$ and the charge density ρ to show that in steady state the normal component of the current density $\mathbf{J}$ at the outer surface of the insulated sphere must vanish. Hint: Integrate the equation of continuity in an infinitesimal cylinder bisected by the boundary, *i.e.,* the pillbox method. What boundary condition does this imply for the normal component of the electric field $\mathbf{E}$ at $r = a$?

c) Use the boundary condition deduced in part b) to determine the equilibrium electrostatic potential $\Phi(r,\theta)$ outside the dielectric shell ($r > a$) using separation of variables in spherical coordinates, *i.e.,* using

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

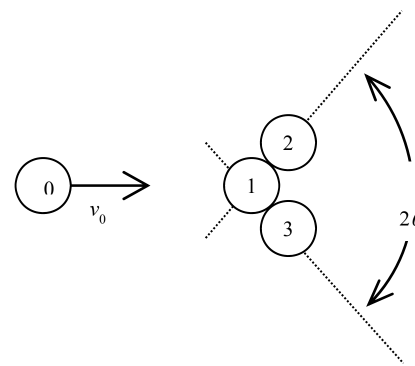
Find the expansion coefficients using the boundary conditions. Note: in this problem $\Phi(r,\theta)$ is not continuous across the interface.

Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, January 4, 2012, 1:00 – 5:00 p.m.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outlines', 'Mathematical Handbook of Formulas and Tables'.

B1. A flat, hard circular disk of radius a and mass m is placed at rest on a flat, horizontal surface on which it is free to slide without friction. Two other identical disks (disks 2 and 3) are placed in contact with the first (disk 1), in such a way that the line connecting the centers of disk 1 and 2 and that connecting the centers of disk 1 and 3 make an angle $\pi > 2\theta > \pi/3$. A fourth disk of the same type (disk 0) moves with initial speed v_0 so that it strikes disk 1 along the line that bisects the angle described above. Find the final speed and direction of motion of each disk after all collisions (assumed to be perfectly elastic) have taken place. (Note: the results will not be altered if you assume that disks 2 and 3 are infinitesimally displaced away from disk 1 along the lines shown.) Briefly describe the qualitatively different outcomes that occur when $\theta > \pi/4$ and $\theta < \pi/4$.

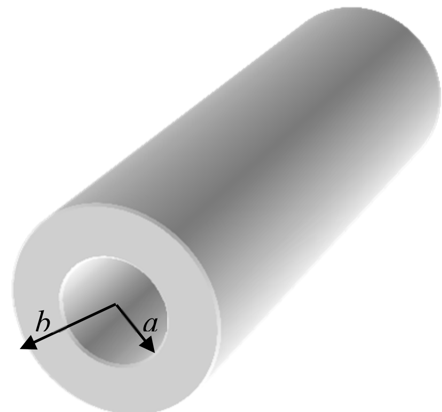


B2. Starting with Newton's second law and the relativistic expression for momentum, derive the relativistic expression for the kinetic energy of the particle. Use the fact that the work done on a particle initially at rest is equal to its kinetic energy. Thus, assume the particle starts from rest at time $t = 0$, and calculate the work done on the particle by a relativistic force. Indicate how the concept of rest-mass energy appears naturally in your calculation.

B3. An anharmonic oscillator has the following Hamiltonian: $H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 + \varepsilon x^4$. If the base states of the oscillator are given by $|n\rangle$ and the raising and lowering operators acting on these base states produce: $a^\dagger|n\rangle = \sqrt{n+1}\;|n+1\rangle$ and $a|n\rangle = \sqrt{n}\;|n-1\rangle$, calculate the energy level shifts to first order using perturbation theory. Hint: x and p can be expressed as

$$x = \sqrt{\frac{\hbar}{m\omega}}\left[\frac{a+a^\dagger}{\sqrt{2}}\right] \text{ and } p = \sqrt{\hbar m\omega}\left[\frac{a-a^\dagger}{i\sqrt{2}}\right].$$

B4. A current flows down a long hollow straight wire of inner radius *a* and outer radius *b*. The hollow wire is made of linear material with magnetic susceptibility χ_m.
The free current density is given by $\vec{J}_f = As^2\,\hat{z}$ for $a < s < b$, where *A* is a constant and the cylindrical coordinates are defined as (s, ϕ, z).



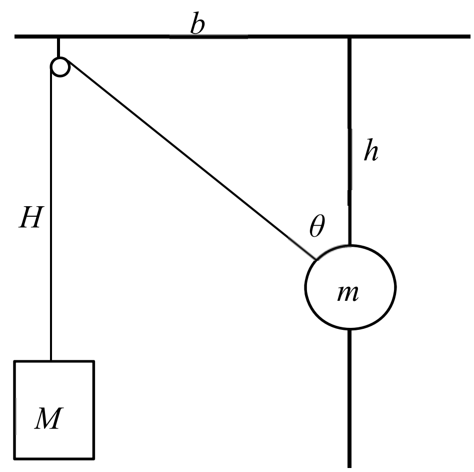
a) Use Ampere's Law in matter to determine the magnetic field $\vec{B}$ in all three regions: inside the cylindrical hole $(s < a)$, inside the cylindrical hollow wire $(a < s < b)$, outside the wire $(s > b)$.

b) Determine the magnetization $\vec{M}$ in all three regions listed above.

c) Determine the volume bound current density $\vec{J}_b$ and the bound surface current $\vec{K}_b$.

d) Calculate the net bound current flowing down the wire.

B5. A ball of mass m (with a small hole through it) is threaded onto a frictionless vertical rod. A massless string of length ℓ is attached to the ball and runs over a massless frictionless pulley and supports a block of mass M. The positions of the two masses can be specified by the angle θ.



a) Write down U(θ) and determine any equilibrium positions for the system.

b) For what values of m and M can equilibrium occur?

c) Determine the stability of any equilibrium positions you find.

B6. The interaction of an atom with a single mode of the electromagnetic field can be described by the Jaynes-Cummings model. It consists of a two-level system with states $|g\rangle$ and $|e\rangle$ coupled to a harmonic oscillator of frequency ω. When the two-level system and the oscillator are in resonance, the Hamiltonian reads $H = H_0 + H_1$ with

$$H_0 = \left(\frac{\hbar\omega}{2}\right)\left[|e\rangle\langle e| - |g\rangle\langle g|\right] + \hbar\omega(a^\dagger a + 1/2)$$

$$H_1 = \left(\frac{\hbar G}{2}\right)\left[|e\rangle a\langle g| + |g\rangle a^\dagger\langle e|\right]$$

Here, a and $a^\dagger$ are the destruction and creation operators of the oscillator quanta, and G is the coupling constant between the atom and the oscillator mode.

a) Find the eigenstates of H_0 and their energies.

b) Assume the system is initially in the state $|\Psi(0)\rangle = |g,n\rangle$, *i.e.*, the atom is in the ground state, and the oscillator is in the *n*-th excited state (*n* quanta). Solve the time-dependent Schroedinger equation for the full Hamiltonian H to find the state $|\Psi(t)\rangle$.

c) You will find oscillations between two basis states, the so-called Rabi oscillations. What is their frequency?