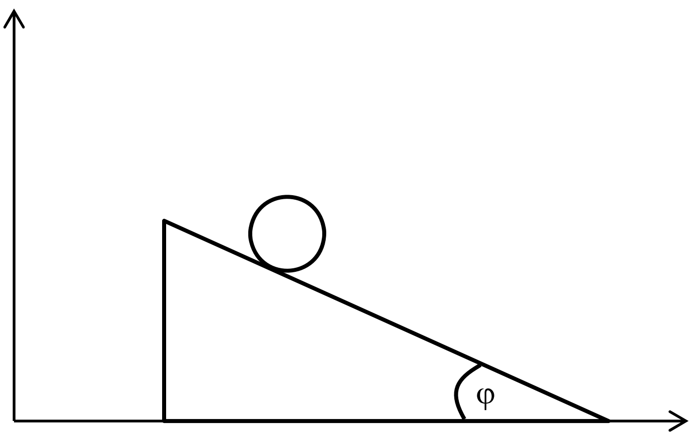
Ph.D. QUALIFYING EXAMINATION – PART A

Tuesday, January 15, 2013, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outlines, 'Mathematical Handbook of Formulas and Tables'.

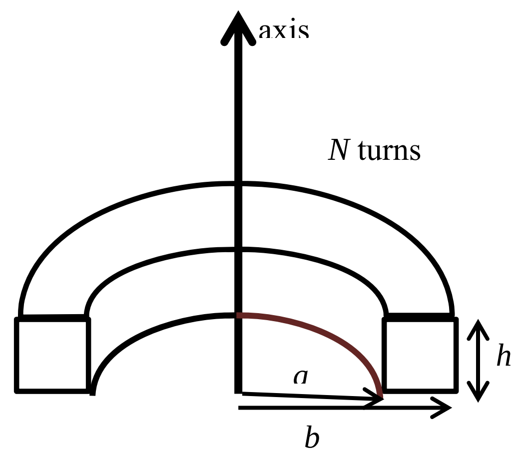
A1. A sphere of mass M radius R rolls without slipping down a triangular block of mass m that is free to move on a frictionless horizontal surface as shown in the figure.



a) Find the Lagrangian for the system (the moment of inertia of a uniform solid sphere about an axis through its center is $I = \frac{2}{5}MR^2$).

b) Find the Lagrangian equations of motion, and solve them assuming the sphere and triangular block are initially at rest, and that the sphere's center is a distance H above the horizontal surface.

A2. a) The figure shows a cut-away side view of a toroidal coil with a rectangular cross section (inner radius a, outer radius b, height h) that carries a total of N closely wound turns. Determine the self-inductance L of the toroidal coil.



A long straight wire runs along the axis of the toroidal coil (like the axle of a wheel). The toroidal coil with self-inductance L and negligible resistance is connected to a resistor R. The current in the long wire as a function of time is given by $I(t) = I_0 e^{-t/\tau}$, where I_0 and τ are constants.

b) Use Faraday's Law and the quasi-static approximation to determine an expression for the emf induced in the toroidal coil and the induced current $I_R(t)$ in the resistor attached to the coil.

c) The current $I_R(t)$ also flows in the coil. This will cause a back emf in the toroidal coil. Determine an expression for the back emf in the coil due to the induced current $I_R(t)$.

d) What is the ratio of this back emf and the "direct" induced emf in part (b)?

A3. In an ionic crystal with reduced mass μ, the net potential energy of an ion is the sum of attractive and repulsive potentials

$$V(r) = -\frac{\alpha e^2}{4\pi\epsilon_0 r} + \lambda e^{-r/\varrho}$$

a) Find the vibrational energy spectrum of the system assuming oscillations are small.

b) Estimate temperature above which these states will be excited.

A4. A particle of mass M and charge e moves freely along a thin nanowire formed into the shape of a **semicircle** of radius a. The wire lies in the top half of the xy-plane, with its center at the origin and its two ends at $x = \pm a$. Focusing only on the particle's angular position along the wire, denote its position wave function by $\psi(\phi)$, where the angle ϕ is measured with respect to the $+x$-axis. Ignore transverse contributions to the energy and consider only the particle's rotational kinetic energy

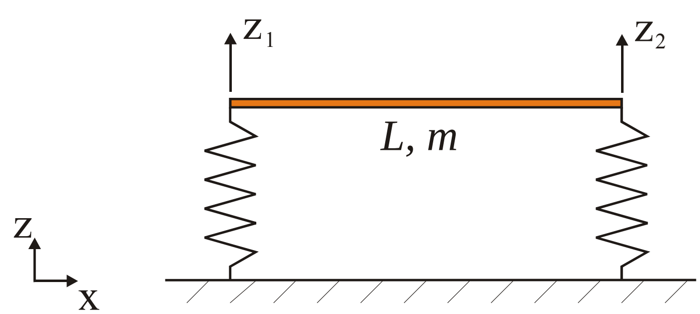
$$H = \frac{L_z^2}{2ma^2} = \frac{\hbar^2 \ell_z^2}{2ma^2} = \epsilon_0 \ell_z^2$$

as it moves along the wire . Here $L_z = \hbar\ell_z = -i\hbar\, \partial/\partial\phi$ is the component of the particle's angular momentum about the z-axis.

(a) Solve the energy eigenvalue equation for this system and, by applying appropriate boundary conditions, (i) determine the energy eigenvalues and their degeneracies, and (ii) construct an orthonormal set of energy eigenfunctions. What is the ground state energy?

(b) Suppose a crystalline defect develops right in the middle of the fragile nano-wire (i.e., at $\phi = \pi/2$). Assume that the defect scatters the particle, via a short range scattering potential $V = v_0\delta(\phi - \pi/2)$. Obtain an expression for the new ground state energy of the system, correct to second order in the scattering strength v_0.

A5. A homogeneous thin rod of length L and mass m rests horizontally on two equal vertical springs of spring constant k (as shown in the figure). The ends of the rod can move independently up and down, i.e., in the z direction. (For small displacements $\Delta z \ll L$, the springs remain vertical during this motion.)



a) Derive the equations of motion for this rod (for small displacements).

b) Find the normal modes and their frequencies. The moment of inertia of a rod about its center is $I = m L^2 / 12$.

A6. In this problem you will be deriving the dispersion relation for an electron in a periodic lattice in a "tight-binding" type calculation. Assume that you have a string of N atoms with the spacing, *b*, between each atom. The string of atoms is wrapped around in a circle, so that atom N+1 is identical to atom 1. Assume there is an extra electron added to the string of atoms and you can label the base states of this electron as $|n\rangle$, where *n* is the specific atom number. If the electron is on the first atom, it is in state $|1\rangle$.

(a) Assume that you can expand the wave function of the electron over the base states $|n\rangle$ as such: $|\psi\rangle = \sum_n c_n(t)|n\rangle$. Using the following notation for the matrix elements of the Hamiltonian, $H_{mn} = \langle m|H|n\rangle$, write down the time-dependent Schrodinger equation and find the set of first order differential equations that describe the time evolution of the $c_n(t)$. You may use the orthonormality of the base states: $\langle n|m\rangle = \delta_{nm}$.

(b) If we assume that the electron cannot hop from atom to atom, we expect that the Hamiltonian is diagonal, with all elements equal to E_0, and the rest zero. This will not describe a moving electron. Instead, we assume that the Hamiltonian additionally can connect nearest neighbor states, say n and $n \pm 1$, with an amplitude of $(-A)$. Write down the matrix representation of the Hamiltonian for N = 6.

(c) We seek stationary-state wave-like solutions to the Schrodinger equation. Take an equation for one of the $\frac{dc_n(t)}{dt} = c_n'(t)$ above and, using the stationary-state time dependence with energy *E*, write $c_n(t) = e^{-i\frac{Et}{\hbar}} a_n$, and simplify the equation to relate the a_n to one another.

(d) If we assume that the atom at position *n* has spatial coordinate $x = bn$, and we expect wavelike solutions (electrons with wave vector *k*), try the solution $a_n = e^{ikx}$, and find the dispersion relation $E = E(k)$ relating the energy to the electron wave vector. Draw a graph of this function.

(e) What range of values can *k* take?

Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, January 16, 2013, 1:00 – 5:00 p.m.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outlines', 'Mathematical Handbook of Formulas and Tables'.

B1. A particle of mass m moves in a potential given by $V(r) = \beta r^k$. Let the angular momentum be L.

a) Find the radius r_0 of the circular orbit.

b) If the particle is given a small kick so that the radius oscillates around r_0 find the frequency ω_r of these small oscillations in r.

c) What is the ratio of the frequency ω_r to the frequency of the (nearly) circular motion $\omega_\theta \equiv \dot{\theta}$?

B2. A relativistic-speed π^0 meson (rest mass m_{π^0}) is traveling with speed v_{π^0} when it decays into two γ-rays making equal angles θ with the direction of motion. Find the angle θ and the energies of the γ-rays.

B3. A sphere of homogenous linear dielectric material is placed in an otherwise uniform electric field, which at large distances from the sphere is directed along the z axis and has magnitude E_0. The dielectric sphere has a radius R and a dielectric constant $\kappa = \varepsilon / \varepsilon_0$, where ε is the permittivity of the sphere and ε_0 is the permittivity of free space.

a) Determine the electric potential both inside and outside the dielectric sphere.

b) Determine the electric field inside the dielectric sphere in terms of κ and E_0.

c) Determine the polarization $\vec{P}$ of the dielectric sphere in terms of ε_0, κ and E_0.

d) Determine the polarization-surface-charge density σ.

Hint: Recall the general solution to Laplace's equation in spherical coordinates if there is no ϕ dependence is given by

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

B4. For times $t < 0$, a particle of mass m is in the ground state of a time-independent quadratic potential $V = \frac{1}{2}m\omega_0^2 x^2$. At $t = 0$ the potential begins to continuously change with time according to the relation $V = \frac{1}{2}m\omega^2(t)x^2$, where the function $\omega(t)$ is constant for $t < 0$, and is given by the relation

$$\omega(t) = \omega_0\left(2 - e^{-t/\tau}\right) \quad \text{for} \quad t > 0.$$

In this expression, τ is a positive constant that determines the time scale over which the potential changes appreciably.

(a) What is the limiting form of the potential as $t \to \infty$? Sketch, on the same graph, the potential (i) for $t < 0$, (ii) for times $t \sim \tau$, and (iii) for times $t \gg \tau$.

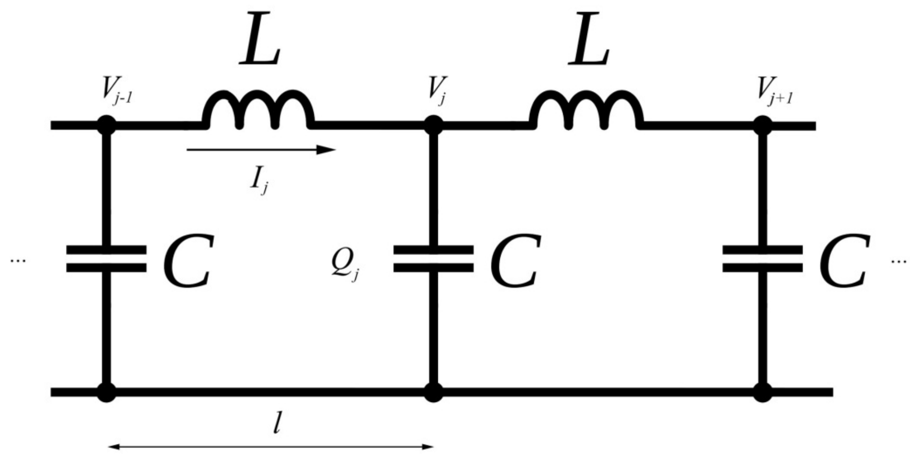
(a) Suppose an energy measurement is made at a time $t \gg \tau$, long after the potential has stopped changing appreciably. Find the probability that the system will be found in the ground state of the Hamiltonian $H(t)$ at the moment this measurement is made, in the limits that (i) the time constant $\tau \gg \omega_0^{-1}$ is much larger than other relevant time scales (the "adiabatic"' limit), and (ii) the time constant $\tau \ll \omega_0^{-1}$ is much smaller than other relevant time scales (the "sudden" limit).

B5. Consider an ideal gas of N atoms of mass m in a cubic box of linear size L at temperature T. Five of the six walls of the box are perfectly elastic (gas particles hitting the wall are elastically reflected). The top wall of the box is cooled down to a very low temperature $T_W << T$ by an external cooling apparatus. Particles hitting this wall temporarily stick to it (losing all their kinetic energy) and are later re-emitted with negligible kinetic energy $k_B T_W$.

Find the amount of heat per unit of time that the cooling apparatus has to remove from the top wall such that it stays at a constant temperature. Formulate your answer in terms of N, L, T, and m.

[Hint: $\int_0^\infty dx\, x^2\, e^{-x} = 2$]

B6. A lossless, infinitely long transmission line (*e.g.* a coaxial cable) has lumped circuit elements as shown in the figure. The potential V_j (a function of time) is the potential at node j relative to the ground.



a) Use Kirchhoff's rules and the definitions of self-inductance and capacitance to show that the potential V_j satisfies the second order differential equation

$$\ddot{V}_j = \frac{1}{LC}\left(V_{j+1} + V_{j-1} - 2V_j\right)$$

b) Assume that V_j is a periodic wave, $V_j = V_0\, e^{i(kjl - \omega t)}$, where $k = 2\pi/\lambda$. What is the dispersion relation for the line? (*i.e.* the relation between ω and λ)

c) Determine the cutoff frequency of the line in terms of L and C. (*i.e.,* the largest allowed frequency)

d) For overhead power lines, the low frequency limit of b) applies. The inductance and capacitance, *per unit length*, are approximately L/l = 2 μH/m and C/l = 10 pF/m, respectively. What is the wave speed in this limit?