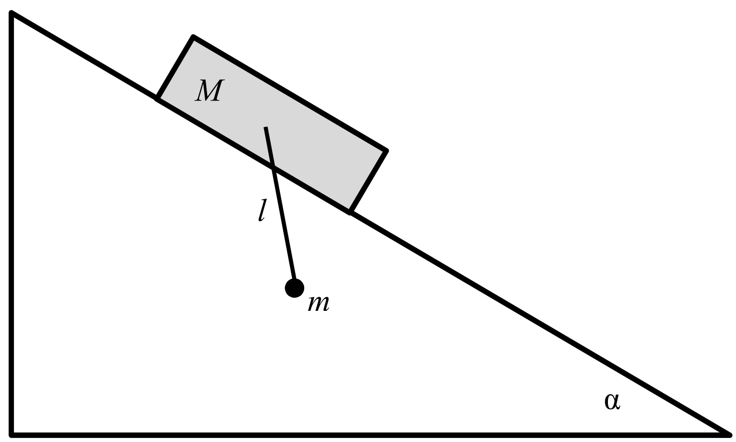
Ph.D. QUALIFYING EXAMINATION – PART A

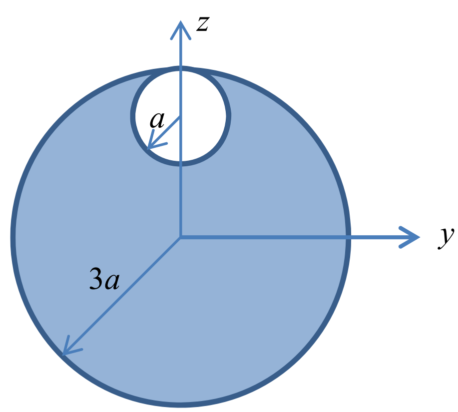
Tuesday, January 12, 2016, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed. Some students find useful the Schaum's outlines, 'Mathematical Handbook of Formulas and Tables'.

A1. A block of mass M slides down a frictionless plane inclined at an angle α. A pendulum of length l and mass m is suspended from mass M as shown in the figure (assume that M hangs slightly over the edge of the incline so the pendulum can hang freely). Find the equations of motion and the equilibrium position of the pendulum.



A2. Consider an insulating sphere of charge of radius $3a$ and uniform charge density ρ with its center at the origin. There is a spherical hole of radius a with its center located on the z axis at $z = 2a$.



a) Use Gauss's Law and the Principle of Superposition to find the electric field everywhere on the positive z axis $(0 \le z \le \infty)$.

b) Determine the electric potential everywhere on the positive z axis $(0 \le z \le \infty)$.

A3. A space station is orbiting a planet of mass M and radius R in a circular orbit of radius $3R$ (purely under the influence of gravity). The astronauts in the space station shoot a probe towards the planet, aiming at its center. What initial speed v_0 (with respect to the space station) do they need to give the probe such that it just grazes the surface of the planet? (Gravity between the space station and the probe can be neglected.) [Hint: Think about conservation laws.]

A4. A particle of mass m is confined to the region $x \in (-a, a)$ by a one-dimensional infinite square well potential of length $2a$. It also interacts with a stationary heavy particle located at the origin, through a short-range attractive potential $V = -v_0\, \delta(x)$.

a) By, e.g., integrating the energy eigenvalue equation across the delta function, show that any energy eigenfunction $\phi(x)$ for this system that does not vanish at $x = 0$ has a discontinuous derivative at that point, and determine the magnitude and sign of the discontinuity

$$\Delta\phi' = \phi'(0^+) - \phi'(0^-).$$

b) Find the most general form of the solution to the energy eigenvalue equation in the regions to the left and to the right of the origin for a "bound-state" of negative energy

$$\varepsilon = -\varepsilon_B.$$

c) Using the condition derived in part (a), and other appropriate boundary conditions, derive a transcendental equation giving the condition under which such a bound state will exist. Show (e.g., graphically) that there will exist such a state provided that the strength v_0 of the interaction exceeds a critical value, and determine, asymptotically, the binding energy in the limit in which v_0 is much larger than that value.

A5. Consider an idealized Sun and Earth as blackbodies in otherwise empty space. The Sun has a surface temperature T_S, and assume that Earth's surface temperature is uniform. The radius of the Earth is R_E, the radius of the Sun is R_S, and the Earth-Sun distance is d.

(a) Considering only the blackbody radiation from the sun and the earth, determine an expression for the temperature of the Earth.

(b) Determine an expression for the total radiation force and the radiation pressure on the Earth.

A6. a) For a well with $V(x) = 0$ for $0 < x < a$, and infinite otherwise, derive the normalized basis functions $\varphi_n(x)$.

b) What are the eigenenergies E_n corresponding to the basis functions?

The variational theorem states that for a wave function $\psi(x)$, the expectation value of the energy is always greater than or equal to the ground state:

$$\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0.$$

c) Calculate the approximate ground state energy for particle in a box using the trial function $\psi(x) = Ax(x - a)$. Normalize the wave function to find A.

d) Suppose you now want to find an approximate wave function and energy for the first excited state. What procedure would you follow? (You do not need to work this out.)

Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, January 13, 2016, 1:00 – 5:00 p.m.

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B1. Two hobos, each of mass m are standing at one end of a stationary railroad car of mass M. The car can move without friction along its tracks. Either hobo can run to the other end of the car and jump off with the same speed u (relative to the car).

a) Find the speed of the recoiling car if the two hobos run and jump from the car simultaneously.

b) Repeat this exercise for the case where the second hobo begins to run and jump only after the first has already jumped. Which case gives the greater speed to the railroad car? ***Hint:*** The speed u is the speed of either hobo *relative to the car* just after he has jumped. It has the same value for each hobo, and is the same in both parts (a) and (b).

B2. The three hydrogen atoms of the boron hydride molecule BH_3 lie on the vertices of an equilateral triangle with the boron atom at the center. An electron bound to this molecule can occupy an orbital on any one of the three hydrogen atoms, in localized quantum states that we will denote by $|1\rangle$, $|2\rangle$, and $|3\rangle$. A non-zero positive matrix element J_0 of the Hamiltonian connects states centered on different orbitals, so that the effective Hamiltonian can be written

$$H = \varepsilon_0 + J_0 \sum_{\langle n,\, m\rangle} (\,|\,n\,\rangle\langle\,m\,| + |\,m\,\rangle\langle\,n\,|\,)$$

in which the sum is over each distinct pair of atoms. A measurement of the energy of the system is taken at a moment when the electron occupies the orbital associated with atom one, i.e. when it is in the state $|\psi\rangle = |1\rangle$.

a) What possible values could be obtained in such a measurement?

b) What is the probability of finding the electron in its lowest energy state?

c) Compute the mean energy $\langle H\rangle$ and the statistical uncertainty ΔH associated with an ensemble of such measurements.

B3. An ideal monatomic gas of N atoms in a volume V is in thermal equilibrium with the temperature T_i. At $t = 0$ all atoms with kinetic energy larger than $\alpha k_B T$, *i.e.*, $\frac{1}{2}mv^2 > \alpha k_B T_i$, are allowed to escape. After that the remaining atoms are assumed to come slowly to a new thermal equilibrium at temperature $T_f(\alpha)$. During the entire process, the system remains isolated from other systems. Note: $\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{1 \bullet 3 \bullet 5 \cdots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}$.

(a) Find an expression for the new equilibrium temperature $T_f(\alpha)$.

(b) Derive an expression for $T_f(\alpha)$ as a function of α for very small $\ll 1$.

B4. A sphere of linear dielectric material with a dielectric constant κ is placed in an otherwise uniform electric field $\vec{E}_0$.

a) Determine the electric potential inside and outside the dielectric sphere.

b) Determine the electric field inside the dielectric sphere.

c) Determine the induced dipole moment $\vec{p}$ inside the dielectric sphere.

d) What is the polarization $\vec{P}$ inside the dielectric sphere.

Recall that if there is no ϕ dependence, the general solution to Laplace's equation for the potential is given as: $V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$.

B5. Consider a vessel that is divided into two parts by a wall. The left part contains a classical ideal gas at temperature T_1, the right part contains an ideal gas at temperature T_2 with $T_2 > T_1$. Both gases consist of the same type of particles of mass m, and the pressures left and right are identical. Now a small hole of area A is opened in the wall separating the two parts.

a) Calculate the net particle current through the hole (difference between the numbers of particles moving from left to right and vice versa per time).

b) Calculate the net energy current (energy transported through the hole per time).

[Hint: Think about which particles hit the hole during a time interval Δt.]

B6. Electromagnetic waves impinging on a good conductor have the singular characteristic of penetrating into the conductor at nearly 90° to its surface *regardless of the angle of incidence*. This makes them an ideal tool to study the structure of a layered material, like the Earth's lithosphere. All that is needed is waves of appropriately low frequencies (for the good–conductor approximation to hold).

a) Use Maxwell's equations to show that the electric field in the conductor satisfies the diffusion equation in this approximation. Assume conductivity σ, and the driving field outside to be sinusoidal; *i.e.* $\propto e^{-i\omega t}$. The diffusion equation is $D\nabla^2\Phi = \partial\Phi / \partial t$. Clearly indicate where you make use of the approximation. Recall: $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \bullet \vec{A}) - \nabla^2 \vec{A}$.

b) Solve the equation by separation of variables assuming $E_y = f(t)\, g(z)$, where t is time and z is distance to the surface. The incident wave has amplitude E_0 at the surface. Make sure the solution is traveling into the conductor; *i.e.* select only a traveling wave solution, and one that travels in the right direction. Show that the amplitude decreases exponentially with scale length δ, the so–called skin depth, and give an explicit expression for δ as a function of both the conductivity σ and the frequency ω.

c) Use Faraday's Law to find the amplitude of the magnetic field at the surface in term of E_0 and δ. [This shows that δ can be obtained from the magnitude of the fields at the surface, and then σ can be determined from δ using the expression you obtained in b)].